EMBEDDINGS AND FACTORIZATIONS OF BANACH SPACES

A Dissertation

by

BENTUO ZHENG

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2007

Major Subject: Mathematics

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Approved by:

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ABSTRACT

Embeddings and Factorizations of Banach Spaces. (August 2007)

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One problem, considered important in Banach space theory since at least the 1970's, asks for intrinsic characterizations of subspaces of a Banach space with an unconditional basis. A more general question is to give necessary and sufficient conditions for operators from L_p $(2 < p < \infty)$ to factor through ℓ_p. In this dissertaion, solutions for the above problems are provided.

More precisely, I prove that for a reflexive Banach space, being a subspace of a reflexive space with an unconditional basis or being a quotient of such a space, is equivalent to having the unconditional tree property. I also show that a bounded linear operator from L_p $(2 < p < \infty)$ factors through ℓ_p if and only it satisfies an upper-(C, p)-tree estimate. Results are then extended to operators from asymptotic l_p spaces.

To my parents and my wife

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CHAPTER I

INTRODUCTION

Banach space theory, as a branch of functional analysis, has been systematically studied since the 1930's. The appearance of Stefan Banach's book [2] is a milestone. The research activity in this area grew rapidly after that. Many fundamental problems were solved and many interesting directions were developed. In this dissertation, I mainly focus on embeddings and factorizations of Banach spaces both of which are well studied and play important roles in Banach space theory.

A. Significance and results of embeddings of Banach spaces

Subspaces of Banach spaces with certain structures, such as Schauder base, unconditional base and so on, turn out to have better behavior. Moreover it is always easier to handle such spaces. So an important and natural task is to determine what spaces embed into superspaces with good structures. In other words, we want to know how these spaces look like and in our language we need to find out the right Banach space conditions they should satisfy.

In 1974, W. J. Davis, T. Figiel, W. B. Johnson and A. Pełczyński [4] proved that a separable dual space embeds into a space with a boundedly complete basis. Later, in 1988, M. Zippin showed that any Banach space with a separable dual (hence separable) embeds into a space with a shrinking basis. This gives a complete characterization of subspaces of spaces with shrinking basis in general.

However, in many problems we want more on the basis structure. In 2005, E. Odell and Th. Schlumprecht proved in [26] that any uniformly convex Banach

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space embeds into a reflexive Banach space with a finite dimensional decomposition satisfying upper-p and lower-q estimates for some $1 < p < q < \infty$. Here a finite dimensional decomposition is a similar but weaker structure than basis which has the same function as basis for most purposes.

There are two major difficulties to get such theorems. One is the discovery of the right intrinsic conditions the statement of which does not require the structure of the containing Banach space. The other is to prove that these conditions, which do not seem to relate to the structure, are actually deeply related to the structure.

In Chapter III (see [16]), we give an instrinsic characterization of subspaces of reflexive spaces with unconditional base. The characterization is that a reflexive space is a subspace of a reflexive space with an unconditional basis if and only if it has the unconditional tree property.

B. Factorizations of Banach spaces

A Banach space X embeds into a Banach space Y if there is an into isomorphism T from X to Y. In particular, $T^{-1}T = id_X$, where id_X is the identity operator on X. Hence X embeds into Y can be stated in the way that the identity operator on X factors through a subspace of Y. A natural extension is to consider a general operator from X to Y and to find the conditions under which the operator factors through Y.

ℓ_p $(1 < p < \infty)$ is certainly one of the most basic and important spaces in Banach space theory. It has a lot of good properties. For example, the canonical basis is symmetric and the space itself is complementably minimal. So when we study operators from spaces which are in some sense "close" to ℓ_p but much more complicated than ℓ_p (such as L_p, asymptotic ℓ_p spaces), we want to see if the operator factors through ℓ_p. If it does, we can then take use of the nice structures of ℓ_p which we know much better. This gives us enough motivation to find out the class of operators from certain spaces which factor through ℓ_p.

In Chapter IV (see [32]), we give a necessary and sufficient condition for an operator from L_p $(2 < p < \infty)$ to factor through ℓ_p. Similar results are extended for operators which factor through c_0.

In Chapter V (see [33]), we consider operators from asymptotic ℓ_p spaces. It is different from the theorems about operators from L_p. In this setting, we find the right conditions for operators from asymptotic ℓ_p spaces to factor through a subspace of an ℓ_p sum of finite dimensional spaces. This still provides a useful connection between asymptotic ℓ_p spaces and ℓ_p.

The last chapter is devoted to some further problems the author is working on.

CHAPTER II

PRELIMINARY MATERIALS

A. Banach spaces and their dual spaces

In this chapter, we record some well known facts that will be used in the sequel. A Banach space is a complete normed vector space. The vector space can be either real or complex. In this dissertation, any Banach space is built on a real vector space, since extension to the complex case is straightforward. By a linear functional on a Banach space X, we mean a linear map from X to $\mathbf{R}$. The dual X^* of X is defined to be the set of all continuous linear functionals on X endowed with the canonical dual norm; $\|f\|^* = \sup_{x \in B_X} |f(x)|$, where B_X denotes the unit ball of X. Under this norm, X^* is a Banach space.

Theorem II.A.1. (Proposition 2.7 in [6]) *Let Y be a closed subspace of a Banach space X. Then $(X/Y)^*$ is isometric to $Y^\perp$, and Y^* is isometric to $X^*/Y^\perp$.*

Here $Y^\perp$ denotes the set of vectors in X^* which take value 0 on Y. Theorem II.A.1 says that the dual of a quotient is a subspace of the dual and the dual of a subspace is a quotient of the dual.

Let X be a Banach space and let X^{**} be the dual of X^*. The canonical embedding π of X into X^{**} is defined for $x \in X$ by

$$\pi(x) : f \mapsto f(x).$$

The Hahn-Banach theorem guarantees that π is a linear into isometry. X is said to be reflexive if π is also onto. In particular, a reflexive space X is isometric to X^{**}.

The converse is not true. There are examples of nonreflexive spaces X which are isometric to X^{**}. But of course the isometry is not the canonical embedding π.

Theorem II.A.2. (Theorem 3.31 and Proposition 3.32 in [6]) *Let X be a Banach space. Then the following are equivalent.*

(i) X is reflexive.

(ii) B_X is weakly compact.

(iii) X^* is reflexive.

If Y is a closed subspace of X, then Y is reflexive.

B. Schauder basis and Schauder finite dimensional decomposition

Let X be an infinite dimensional Banach space. A sequence $(e_n)_{n=1}^{\infty}$ is a Schauder basis of X if for every $x \in X$, there is a unique sequence of real numbers (a_n) so that $\sum_{i=1}^{n} a_i e_i$ converges to x in norm. In this case, we write $x = \sum_{i=1}^{\infty} a_i e_i$.

1. Basis constant

If (e_i) is a Schauder basis of X, then the canonical projections $P_n : X \to X$ are defined by $P_n(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=1}^{n} a_i e_i$. One can show that (P_n) satisfy the following property:

(i) $\dim(P_n(X)) = n$;

(ii) $P_n P_m = P_m P_n = P_{\min(m,n)}$;

(iii) $P_n(x)$ converges to x for every $x \in X$.

Conversely, if there are a sequence of bounded linear projections (P_n) on X which satisfy (i), (ii) and (iii), then (P_n) are canonical projections associated with some Schauder basis of X. The basis constant is defined to be the supremum of the norms of P_n. A Schauder basis (e_i) is called normalized if $\|e_i\| = 1$ for all $i \in \mathbf{N}$. It is called monotone if the basis constant of (e_i) is 1. In this situation, $\|P_n\| = 1$ for every n (it is obvious that $\|P_n\| \geq 1$). A sequence (x_i) in X is called a basic sequence if (x_i) is a Schauder basis for its closed linear span. The following theorem was proved by Banach.

Theorem II.B.1. (Proposition 6.13 in [6]) *Let (e_i) be a sequence of nonzero vectors in a Banach space X. (e_i) is a basic sequence if and only if there is a $K > 0$ such that for all $n < m$ and scalars $a_1, a_2, ..., a_m$, we have*

$$\|\sum_{i=1}^{n} a_i e_i\| \leq K \|\sum_{i=1}^{m} a_i e_i\|.$$

Moreover, the smallest such constant K is equal to the basis constant of (e_i).

2. Shrinking basis and boundedly complete basis

Let (e_i) be a Schauder basis of X. We say $(e_i^*) \subset X^*$ are the biorthogonal functionals of (e_i) if for $i, j \in \mathbf{N}$, $e_i^*(e_j) = \delta_i(j)$. The Schauder basis (e_i) is called shrinking if the closed linear span of (e_i^*) is X^*. It is called boundedly complete if $\sum a_i e_i$ converges whenever $\sup_n \|\sum_{i=1}^n a_i e_i\|$ is finite. The following is a theorem of James.

Theorem II.B.2. *Let X be a Banach space with a Schauder basis (e_i). Then X is reflexive if and only if (e_i) is both shrinking and boundedly complete.*

3. Schauder finite dimensional decomposition

Let X be an infinite dimensional Banach space. An infinite sequence of finite dimensional spaces $(E_i) \subset X$ is called a Schauder finite dimensional decomposition (FDD) of X if every $x \in X$ has a unique representation of the form $x = \sum_{i=1}^{\infty} x_i$ with $x_i \in E_i$. Let P_j be the natural projection from X onto E_j and let E_j^* be the dual of E_j. Since P_j is a projection, the adjoint operator P_j^* is an isomorphic embedding from E_j^* into X^*. In other words, $P_j^* E_j^*$ is a subspace of X^*. Similar to Schauder basis, an FDD (E_i) of X is shrinking if $(P_j^* E_j^*)$ is an FDD for X^*. It is boundedly complete if $\sum x_i$ converges whenever $x_i \in E_i$ and $\sup_n \| \sum_{i=1}^n x_i \|$ is finite. It is clear that when all the E_i's are one dimensional spaces, then unit vectors (e_i) with $e_i \in E_i$ form a Schauder basis of X. Hence the concept FDD is a generalization of basis. We also have the following theorem which is a generalization of Theorem II.B.2.

Theorem II.B.3. *Let X be a Banach space with an FDD (E_i). Then X is reflexive if and only if (E_i) is both shrinking and boundedly complete.*

4. Unconditional basis and unconditional finite dimensional decomposition

Let X be a Banach space with a Schauder basis (e_i). We say that (e_i) is unconditional if there exists a $C > 0$ so that for any $n \in \mathbf{N}$ and any choice of signs $(\theta_i)_{i=1}^n$,

$$\| \sum_{i=1}^{n} \theta_i a_i e_i \| \leq C \| \sum_{i=1}^{n} a_i e_i \|.$$

A sequence of vectors (x_i) in X are called unconditional basic if they form an unconditional basis for their closed linear span.

Theorem II.B.4. (Proposition 1.c.6 in [20]) *A basic sequence (x_i) is unconditional* *if and only if any of the following conditions holds.*

(i) *For every permutation π of the positive integers, the sequence $(x_{\pi(n)})$ is a basic sequence*

(ii) *For every subset σ of the positive integers, the convergence of $\sum_{i=1}^{\infty} a_i x_i$ implies the convergence of $\sum_{i\in\sigma} a_i x_i$*

(iii) *The convergence of $\sum_{i=1}^{\infty} a_i x_i$ implies the convergence of $\sum b_i x_i$ whenever $|b_i| \leq |a_i|$ for all i.*

A sequence $(x_i) \subset X$ converges unconditionally if the series $\sum \theta_i x_i$ converges for every choice of signs θ_i. A sequence of finite dimensional spaces $(E_i) \subset X$ is called an unconditional finite dimensional decomposition (UFDD) of a Banach space X if (E_i) is an FDD for X and for every $x \in X$, the representation $\sum x_i$ of x with $x_i \in E_i$, converges unconditionally.

C. Isomorphisms and Banach Mazur distance

Let X, Y be two Banach spaces. A map $T : X \to Y$ is a bounded linear operator if it is linear and norm to norm continuous. X is said to be isomorphic to Y if there exists an one-to-one bounded linear operator T from X onto Y. In this case, we say T is an isomorphism. If the bounded linear operator T is an isomorphism onto a subspace of Y, then we say X embeds into Y or X is isomorphic to a subspace of Y. Under this definition, any two n-dimensional spaces are isomorphic. But there are uncountablely many infinite dimensional separable Banach spaces which are mutually nonisomorphic. In the isomorphic theory, we do not distinguish two Banach spaces which are isomorphic. If there is a norm one isomorphism T from X to Y so that the norm of T^{-1} is also one, then we say X and Y are isometric.

The Banach-Mazur distance between X and Y is defined to be the infimum of $\|T\|\|T^{-1}\|$, where T ranges through all isomorphisms from X to Y. So if X and Y are isometric, then the Banach-Mazur distance of X and Y is 1.

D. Trees and branches in Banach spaces

A tree in a Banach space X is a family $(x_A)_{A\in[\mathbf{N}]^{<\omega}} \subset X$ indexed over all finite subsets of $\mathbf{N}$. Let $A = \{n_1, ..., n_m\}$ with $n_1 < ... < n_m$ and $B = \{j_1, ..., j_r\}$ with $j_1 < ... < j_r$. We say A is an initial segment of B if $m \leq r$ and $n_i = j_i$ when $1 \leq i \leq m$. The tree order on $(x_A)_{A\in[\mathbf{N}]^{<\omega}}$ is given by $x_A \leq x_B$ if A is an initial segment of B. A branch of a tree is a maximal linearly ordered subset of the tree under the tree order. A tree is called a normalized weakly null tree if for every $A \in [\mathbf{N}]^{<\omega}$, the sequence $(x_{\{n\}\cup A})_{n>\max A}$ is weakly null and $\|x_A\| = 1$.

Many Banach space conditions appear in the way that every normalized weakly null sequence (x_n) in X has a subsequence with some property (P). Sometimes this is the right hypothesis to conclude that X has some property (Q). If we define a tree by letting $x_A = x_{\max A}$, then the set of branches of the tree (x_A) is exactly the set of subsequences of (x_n). Hence, the condition that every normalized weakly null tree has a branch with property (P) is stronger than the condition that every normalized weakly null sequence has a subsequence with property (P). In some cases, these two conditions are equivalent. For example, when considering subspaces of L_p $(1 < p < \infty)$, the condition that every normalized weakly null tree has a branch which is C-equivalent to the unit vector basis of ℓ_p is equivalent to the condition that every normalized weakly null sequence has a subsequence which is C-equivalent to the unit vector basis of ℓ_p. But in general, these two conditions are not equivalent. There is a counterexample of E. Odell and Th. Schlumprecht [25] which admits the above subsequence condition but not the above tree condition.

E. Block basis and blockings of FDD

Given an FDD (E_n) of X, (x_n) is said to be a block sequence w.r.t. (E_n) if there exists a sequence of integers $0 = m_1 < m_2 < m_3 < \ldots$ such that $x_n \in \bigoplus_{j=m_n+1}^{m_{n+1}} E_j, \forall n \in \mathbf{N}$.

(x_n) is said to be a skipped-block sequence w.r.t. (E_n) if there exists a sequence of increasing integers $0 = m_1 < m_2 < m_3 < \ldots$ such that $m_n + 1 < m_{n+1}$ and $x_n \in \bigoplus_{j=m_n+1}^{m_{n+1}-1} E_j, \forall n \in \mathbf{N}$.

Let $\delta = (\delta_i)$ be a sequence of positive numbers decreasing to 0. We say (y_n) is a δ-skipped block with respect to (E_n) if there is a skipped-block sequence (x_n) so that $\|y_n - x_n\| \leq \delta_n \|y_n\|$ for all $n \in \mathbf{N}$. We say (F_n) is a blocking of (E_n) if there is a sequence of increasing integers $0 = k_0 < k_1 < \ldots$ so that $F_n = \oplus_{j=k_{n-1}+1}^{k_n} E_j$.

Let $0 = k_1 < k_2 < \ldots$ be an increasing sequence of integers and put $F_n = \oplus_{i=k_n+1}^{k_{n+1}} E_i$ for all $n \in \mathbf{N}$. Then the decomposition (F_n) of X is said to be a blocking of the decomposition (E_n).

CHAPTER III

A CHARACTERIZATION OF SUBSPACES OF SPACES WITH UNCONDITIONAL BASES

A. Introduction

It has long been known that Banach spaces with unconditional bases as well as their subspaces are much better behaved than general Banach spaces, and that many of the reflexive spaces (including $L_p(0,1), 1 < p < \infty$) that arise naturally in analysis have unconditional bases. It is however difficult to determine whether a given Banach space has an unconditional basis or embeds into a space which has an unconditional basis. Two problems, considered important since at least the 1970's, stand out.

(a) Give an intrinsic condition on a Banach space X which is equivalent to the embeddability of X into a space with an unconditional basis.

(b) Does every complemented subspace of a space with an unconditional basis have an unconditional basis?

Problem (b) remains open, but in this paper we provide a solution to problem (a) for reflexive Banach spaces. This characterization also yields that a quotient of a reflexive space with an unconditional basis embeds into a reflexive space with unconditional basis, which solves another problem from the 1970's. Here some condition on the space with an unconditional basis is needed because every separable Banach space is a quotient of l_1.

There is, of course, quite a lot known around problems (a) and (b). For example, Pełczyński and Wojtaszczyk [29] proved that if X has an unconditional expansion of identity (i.e., a sequence (T_n) of finite rank operators such that $\sum T_n$ converges unconditionally in the strong operator topology to the identity on X), then X is isomorphic to a complemented subspace of a space that has an unconditional finite dimensional decomposition (UFDD). Later, Lindenstrauss and Tzafriri [20] showed that every space with an UFDD embeds (not necessarily complementably) into a space with an unconditional basis. As regards reflexive spaces, it was shown in [7] using a result from [4] (and answering a question from that paper) that if a reflexive Banach space embeds into a space with an unconditional basis, then it embeds into a reflexive space with an unconditional basis. As regards the quotient problem we mentioned above, Feder [5] gave a partial solution by proving that if X is a quotient of a reflexive space which has an UFDD and X has the approximation property, then X embeds into a space with an unconditional basis.

It is well known and easy to see that if a Banach space X embeds into a space with an unconditional basis, then X has the unconditional subsequence property; that is, there exists a $K > 0$ so that every normalized weakly null sequence in X has a subsequence which is K-unconditional. In fact, failure of the unconditional subsequence property is the only known criterion for proving that a given reflexive space does not embed into a space with an unconditional basis. However, in the last section we construct a Banach space which has the unconditional subsequence property but does not embed into a Banach space that has an unconditional basis. This is not surprising, given previous examples of E. Odell and Th. Schlumprecht [25]. Moreover, Odell and Schlumprecht have taught us that by replacing a subsequence property by the corresponding "branch of a tree" property, you get a stronger property that sometimes can be used to give a characterization of spaces that embed into a space with some kind of structure. The property relevant for us is the unconditional tree property and Odell and Schlumprecht's beautiful results are essential tools for us. We use standard Banach space theory terminology, as can be found in [20].

B. Main results

Definition III.B.1. We say X has the C-UTP if every normalized weakly null tree in X has a C-unconditional branch for some $C > 0$. X has the UTP if X has the C-UTP for some $C > 0$.

Remark III.B.2. E. Odell, Th. Schlumprecht and A. Zsak proved in [27] that if every normalized weakly null tree in X admits a branch which is unconditional, then X has the C-UTP for some $C > 0$. A simpler proof will appear in the forthcoming paper of R. Haydon, E. Odell and Th. Schlumprecht [10]. So there is no ambiguity when using the term "UTP".

Definition III.B.3. Let X be a Banach space with an FDD (E_n). If there exists a $C > 0$ so that every skipped block sequence with respect to (E_n) is C-unconditional, then we say (E_n) is an unconditional skipped blocked FDD (USB FDD).

The following is a blocking lemma of W. B. Johnson and M. Zippin (see [18] or Proposition 1.g.4(a) in [20]) which will be used later.

Lemma III.B.4. *Let $T : X \to Y$ be a bounded linear operator. Let (B_n) be a shrinking FDD of X and let (C_n) be an FDD of Y. Let (δ_n) be a sequence of positive numbers tending to 0. Then there are blockings $(B_n^{'})$ of (B_n) and $(C_n^{'})$ of (C_n) so that, for every $x \in B_n^{'}$, there is a $y \in C_{n-1}^{'} \bigoplus C_n^{'}$ such that $\|Tx - y\| \leq \delta_n \|x\|$.*

The lemma above actually works for any further blockings of $(B_n^{'})$ and $(C_n^{'})$. To be more precise, we have the following stronger result which is actually a formal consequence of Lemma III.B.4 as stated.

Lemma III.B.5. *Let $T : X \to Y$ be a bounded linear operator. Let (B_n) be a shrinking FDD of X and let (C_n) be an FDD of Y. Let (δ_n) be a sequence of positive numbers tending to 0. Then there are blockings $(B_n^{'})$ of (B_n) and $(C_n^{'})$ of (C_n) so that, for any further blockings $(\tilde{B}_n)$ of $(B_n^{'})$ with $\tilde{B}_n = \oplus_{i=k_n}^{k_{n+1}-1} B_i^{'}$ and $(\tilde{C}_n)$ of $(C_n^{'})$ with $\tilde{C}_n = \oplus_{i=k_n}^{k_{n+1}-1} C_i^{'}$ and for any $x \in \tilde{B}_n$, there is a $y \in \tilde{C_{n-1}} \oplus \tilde{C}_n$ such that $\|Tx - y\| \leq \delta_n \|x\|$.*

Proof. Let (δ_i) be a sequence of positive numbers decreasing to 0. Let $(\tilde{\delta}_i)$ be another sequence of positive numbers which go to 0 so fast that $\sum_{j=i}^{\infty} \tilde{\delta}_j < \delta_i/2\lambda$, where λ is the basis constant for (B_n). By Lemma III.B.4, we get blockings $(B_n^{'})$ of (B_n) and $(C_n^{'})$ of (C_n) so that for every $x \in B_n^{'}$, there is a $y \in C_{n-1}^{'} \oplus C_n^{'}$ such that $\|Tx - y\| \leq \tilde{\delta}_n \|x\|$. Let $\tilde{B}_n = \oplus_{i=k_n}^{k_{n+1}-1} B_i^{'}$ and $\tilde{C}_n = \oplus_{i=k_n}^{k_{n+1}-1} C_i^{'}$ be blockings of $(B_n^{'})$ and $(C_n^{'})$. Let $x \in \tilde{B}_n$. Then we can write $x = \sum_{i=k_n}^{k_{n+1}-1} x_i, x_i \in B_i^{'}$. So by our construction of $(B_n^{'})$ and $(C_n^{'})$, there are $y_i \in C_{i-1}^{'} \oplus C_i^{'}, k_n \leq i \leq k_{n+1} - 1$ so that $\|Tx_i - y_i\| \leq \tilde{\delta}_i \|x_i\|, k_n \leq i \leq k_{n+1} - 1$. Let $y = \sum_{i=k_n}^{k_{n+1}-1} y_i \in \tilde{C_{n-1}^{'}} \oplus \tilde{C}_n^{'}$. Then we have

$$\|Tx - y\| \leq \sum_{i=k_n}^{k_{n+1}-1} \tilde{\delta}_i \|x_i\| \leq \sum_{i=k_n}^{k_{n+1}-1} 2\lambda \tilde{\delta}_i \|x\| \leq \delta_n \|x\|.$$

The following convenient reformulation of Lemma III.B.4 will also be used (see [17] and [18] or [24]).

Lemma III.B.6. *Let $T : X \mapsto Y$ be a bounded linear operator. Let (B_n) be a shrinking FDD for X and (C_n) be an FDD for Y. Let (δ_i) be a sequence of positive numbers decreasing to 0. Then there is a blocking $(B_n^{\prime})$ of (B_n) and a blocking $(C_n^{\prime})$*

of (C_n) *so that for any* $x \in B'_n$ *and any* $m \neq n, n-1$,

$$\|Q_m(Tx)\| < \delta_{\max\{m,n\}}\|x\|,$$

where Q_j *is the canonical projection from* Y *onto* C'_j.

Remark III.B.7. The qualitative content of Lemma III.B.6 is that there are blockings (B'_n) of (B_n) and (C'_n) of (C_n) so that TB'_n is essentially contained in $C'_{n-1} + C'_n$.

Our first theorem says that the unconditional tree property for reflexive Banach spaces passes to quotients. It plays a key role in this paper and involves the lemmas above as well as results and ideas of Odell and Schlumprecht.

Let us explain the sketch of the proof of the special case when Y is a reflexive space with the UTP and Y has an FDD (E_n), while X is a quotient of Y which has an FDD (V_n). Since Y has the UTP, by Odell and Schlumprecht's fundamental result [25], there is a blocking (F_n) of the (E_n) which is an USB FDD. Then we use the "killing the overlap" technique of [12] to get a further blocking (G_n) so that any norm one vector y is a small perturbation of the sum of a skipped block sequence (y_i) with respect to (F_n) and $y_i \in G_{i-1} \oplus G_i$. Let $Q : Y \mapsto X$ be the quotient map. Using Lemma III.B.5 and passing to a further blocking, without loss of generality, we assume that QG_i is essentially contained in $H_{i-1} + H_i$, where (H_i) is the corresponding blocking of (V_n). Let (x_A) be a normalized weakly null tree in X. We then choose a branch (x_{A_i}) so lacunary that (x_{A_i}) is a small perturbation of a block sequence of (H_n) and for each i, there is at least one H_{k_i} between the essential support of x_{A_i} and $x_{A_{i+1}}$. Let $x = \sum a_i x_{A_i}$ with $\|x\| = 1$. Considering a preimage y of x under the quotient Q from Y onto X (with $\|y\| = 1$), by our construction, we can essentially write y as the sum of (y_i) where (y_i) is a skipped block sequence with respect to (F_n). Since (F_n) is USB, (y_i) is unconditional. By passing to a suitable blocking (z_i) of (y_i) and using Lemma III.B.5, it is not hard to show that Qz_i is essentially equal to $a_i x_{A_i}$. Noticing that (z_i) is also unconditional, we conclude that (x_{A_i}) is also unconditional.

For the general case when X and Y do not have an FDD, we have to embed them into some superspaces with FDD. The difficulty is that when we decompose a vector in Y as the sum of disjointly supported vectors in the superspace, we do not know that the summands are in Y. The same problem occurs for vectors in X. This makes the proof rather technical and a lot of computations appear.

Theorem III.B.8. *Let X be a quotient of a separable reflexive Banach space Y with UTP. Then X has UTP.*

Proof. By Zippin's result [34], Y embeds isometrically into a reflexive space Z with an FDD. A key point in the proof is that Odell and Schlumprecht proved (Proposition 2.4 in [26]) that there is a further blocking (G_n) of the FDD for Z, $\delta = (\delta_i)$ and a $C > 0$ so that every δ-skipped block sequence $(y_i) \subset Y$ with respect to (G_i) is C-unconditional. Let λ be the basis constant for (G_n).

Since X is separable, we can regard X as a subspace of L_∞. Let $\epsilon > 0$. We may assume that

(a) $\sum_{j>i} \delta_j < \delta_i$,

(b) $i\delta_i < \delta_{i-1}$,

(c) $\sum \delta_i < \epsilon$.

Let Q be a quotient map from Y onto X, which can be extended to a norm one map from Z into L_∞ and we still denote it by Q. QZ, as any separable subspace of

L_∞, is contained in some super space isometric to $C(\Delta)$ with monotone basis (v_i). Here Δ is the Cantor set.

Let (x_A) be a normalized weakly null tree in X. Then we let (E_n) and (F_n) be blockings of (G_i) and (v_i) respectively which satisfy the conclusions of Lemma III.B.5 and Lemma III.B.6. Using the "killing the overlap" technique (see Proposition 2.6 in [26]), we can find a further blocking ($\tilde{E_n} = \oplus_{i=l(n)+1}^{l(n+1)} E_i$) so that for every $y \in S_Y$, there exists $(y_i) \subset Y$ and integers (t_i) with $l(i-1) < t_i \leq l(i)$ for all i such that

(I) $y = \sum y_i$,

(II) For $i \in \mathbf{N}$, either $\|y_i\| < \delta_i$ or $\|\sum_{j=t_{i-1}+1}^{t_i-1} P_j y_i - y_i\| < \delta_i \|y_i\|$,

(III) $\|\sum_{j=t_{i-1}+1}^{t_i-1} P_j y - y_i\| < \delta_i$,

(IV) $\|P_{t_i} y\| < \delta_i$ for $i \in \mathbf{N}$,

where P_j is the canonical projection from Y onto E_j. Let $\tilde{F_n} = \oplus_{i=l(n)+1}^{l(n+1)} F_i$ and let $\tilde{P_j}$ be the canonical projection from X onto $\tilde{F_j}$. Since (x_A) is a weakly null tree, we can pick inductively a branch (x_{A_i}) and an increasing sequence of integers $1 = k_0 < k_1 < \ldots$ such that for any $i \in \mathbf{N}$

(i) $\|\sum_{j=k_{2i-2}}^{k_{2i-1}-1} \tilde{P_j} x_{A_i} - x_{A_i}\| < \delta_i$,

(ii) $\|\sum_{j=k_{2i-2}}^{k_{2i-1}-1} \tilde{P_j} x_{A_t}\| < \delta_{\max\{i,t\}}$, for any $t \neq i$.

We will prove that (x_{A_i}) is unconditional. Let $x = \sum a_i x_{A_i}$, $\|x\| = 1$. Let $y \in S_Y$ so that $Q(y) = x$. Then y can be written as $\sum y_j$ where (y_j) satisfy (I), (II), (III) and (IV). Define $k_{-1} = -1$ and let $z_i = \sum_{j=k_{2i-3}+2}^{k_{2i-1}+1} y_j$. We will prove that $\|Qz_i - a_i x_{A_i}\|$ is small.

$$
\begin{aligned}
\|Qz_i - a_i x_{A_i}\| &\leq \|Q(\sum_{j=t_{k_{2i-3}+1}}^{t_{k_{2i-1}+1}} P_j y) - (\sum_{j=k_{2i-2}}^{k_{2i-1}-1} \tilde{P}_j)x\| \\
&+ \|z_i - \sum_{j=t_{k_{2i-3}+1}}^{t_{k_{2i-1}+1}} P_j y\| \\
&+ \|a_i x_{A_i} - (\sum_{j=k_{2i-2}}^{k_{2i-1}-1} \tilde{P}_j)x\|.
\end{aligned}
$$

Hence we need to estimate the three terms in the right hand side of the above inequality. By the construction, for $i > 1$, we have

$$
\begin{aligned}
\|z_i - \sum_{j=t_{k_{2i-3}+1}}^{t_{k_{2i-1}+1}} P_j y\| &< \sum_{j=k_{2i-3}+2}^{k_{2i-1}+1} (\|\sum_{l=t_{j-1}+1}^{t_j-1} P_l y - y_j\| + \|P_{t_{j-1}} y\|) + \|P_{t_{k_{2i-1}+1}} y\| \\
&< \sum_{j=k_{2i-3}+2}^{k_{2i-1}+1} \delta_j + \sum_{j=k_{2i-3}+2}^{k_{2i-1}+2} \delta_{j-1} \\
&< \delta_{k_{2i-3}+1} + \delta_{k_{2i-3}} \\
&< \delta_i.
\end{aligned}
$$

By direct calculation, for $i = 1$, we have

$$
\|z_1 - \sum_{j=1}^{t_{k_1+1}} P_j y\| < 2\delta_1.
$$

This gives an estimate of the second term. For the third term, we have

$$
\begin{aligned}
\|a_i x_{A_i} - (\sum_{j=k_{2i-2}}^{k_{2i-1}-1} \tilde{P}_j)x\| &< \|(\sum_{j=k_{2i-2}}^{k_{2i-1}-1} \tilde{P}_j)(a_i x_{A_i} - x)\| + \|a_i(x_{A_i} - (\sum_{j=k_{2i-2}}^{k_{2i-1}-1} \tilde{P}_j)x_{A_i}\| \\
&< 2(k_{2i-2}\delta_{k_{2i-2}} + \sum_{j \geq k_{2i-1}} \delta_j) + 2\delta_i \\
&< 2(\delta_{k_{2i-2}-1} + \delta_{k_{2i-1}-1}) + 2\delta_i \\
&< 4\delta_i.
\end{aligned}
$$

For the first term, let Q_j be the canonical projection from X onto F_j and let $J_1 =$

$[t_{k_{2i-3}+1}, t_{k_{2i-1}+1}]$, $J_2 = [l_{k_{2i-2}}+1, l_{k_{2i-1}}]$ and $J_1' = (t_{k_{2i-3}+1}, t_{k_{2i-1}+1})$. Then we have

$$\begin{aligned}
\|Q(\sum_{j\in J_1} P_j y) - (\sum_{j\in J_2} Q_j)Qy\| &\leq \|Q(\sum_{j\in J_1} P_j y) - (\sum_{j\in J_1} Q_j)Qy\| + \|(\sum_{j\in J_1} Q_j)Qy - (\sum_{j\in J_2} Q_j)Qy\| \\
&= \|Q(\sum_{j\in J_1} P_j y) - (\sum_{j\in J_1} Q_j)Qy\| + \|\sum_{j\in J_1 - J_2} Q_j(\sum a_i x_{A_i})\| \\
&< \|Q(\sum_{j\in J_1} P_j y) - (\sum_{j\in J_1} Q_j)Qy\| + 4\delta_i \\
&\leq \|(\sum_{j\notin J_1} Q_j)Q(\sum_{j\in J_1} P_j y)\| + \|(\sum_{j\in J_1} Q_j)Q(\sum_{j\notin J_1} P_j y)\| + 4\delta_i \\
&< \|(\sum_{j\notin J_1} Q_j)Q(\sum_{j\in J_1'} P_j y)\| + \|(\sum_{j\in J_1'} Q_j)Q(\sum_{j\notin J_1} P_j y)\| + 6\delta_i \\
&< 2\lambda\delta_i + 2\lambda\delta_i + 6\delta_i \\
&= (4\lambda + 6)\delta_i.
\end{aligned}$$

From the Inequalities above, we conclude that

$$\|Qz_i - a_i x_{A_i}\| < (4\lambda + 12)\delta_i.$$

Let $(\epsilon_i) \subset \{-1, 1\}^{\mathbf{N}}$. Let $I \subset \mathbf{N}$ be the set of indices $i \in \mathbf{N}$ for which $\|y_i\| < \delta_i$ and let $I_i = [k_{2i-3}+2, k_{2i-1}+1]$. So $z_i = \sum_{j\in I_i} y_j$. Let $z_i' = \sum_{j\in I_i - I} y_j$. It is easy to verify that $\|z_i - z_i'\| < \delta_i$. Hence $\|Qz_i' - a_i x_{A_i}\| < (4\lambda + 13)\delta_i$. Now by (II), we know that (z_i') is a δ-skipped block sequence. Hence, (z_i') is unconditional. So we have

$$\begin{aligned}
\|\sum \epsilon_i a_i x_{A_i}\| &\leq \|Q(\sum \epsilon_i z_i')\| + (4\lambda + 13)\epsilon \\
&\leq C\|\sum z_i'\| + (4\lambda + 13)\epsilon \\
&< C(\|\sum z_i\| + \sum \delta_i) + (4\lambda + 13)\epsilon \\
&\leq C + (C + 4\lambda + 13)\epsilon,
\end{aligned}$$

This shows (x_{A_i}) is an unconditional sequence.

The following is an elementary lemma which will be used later. We omit the standard proof.

Lemma III.B.9. *Let X be a Banach space and X_1, X_2 be two closed subspace of X. If $X_1 \cap X_2 = \{0\}$ and $X_1 + X_2$ is closed, then X embeds into $X/X_1 \oplus X/X_2$.*

In [15], W. B. Johnson and H. P. Rosenthal proved that any separable Banach space X admits a subspace Y so that both Y and X/Y have a FDD. The proof uses Markuschevich bases. A Markuschevich basis for a separable Banach space X is a biorthogonal system $\{x_n, x_n^*\}_{n\in\mathbf{N}}$ for which the span of the x_n's is dense in X and the x_n^*'s separate the points of X. By Theorem 1.f.4 in [20], every separable Banach space X has a Markuschevich basis $\{x_n, x_n^*\}_{n\in\mathbf{N}}$ so that $[x_n^*]$ contains any designated separable subspace of X^*. The following lemma is a stronger form of the result of Johnson and Rosenthal which follows from the original proof. For the convenience of the readers, we give a sketch of the proof. We use $[x_i]_{i\in I}$ to denote the closed linear span of $(x_i)_{i\in I}$.

Lemma III.B.10. *Let X be a separable Banach space. Then there exists a subspace Y with FDD (E_n) so that for any blocking (F_n) of (E_n) and for any sequence $(n_k) \subset \mathbf{N}$, $X/\overline{span\{(F_{n_k})_{k=1}^{\infty}\}}$ admits an FDD (G_n). Moreover, if X^* is separable, (E_n) and (G_n) can be chosen to be shrinking.*

Proof. Let $\{x_i, x_i^*\}$ be a Markuschevich basis for X so that $[x_i^*]$ is a norm determining subspace of X^* and even $[x_i^*] = X^*$ if X^* is separable. Then we can choose inductively finite sets $\sigma_1 \subset \sigma_2 \subset ...$ and $\eta_1 \subset \eta_2 \subset ...$ so that $\sigma = \bigcup_{n=1}^{\infty} \sigma_n$ and $\eta = \bigcup_{n=1}^{\infty} \eta_n$ are complementary infinite subsets of the positive integers and for $n = 1, 2, ...,$

(i) if $x^* \in [x_i^*]_{i\in\eta_n}$, there is a $x \in [x_i]_{i\in\eta_n \bigcup \sigma_{n+1}}$ so that $\|x\| = 1$ and $|x^*(x)| > (1 - \frac{1}{n+1})\|x^*\|$;

(ii) if $x \in [x_i]_{i\in\sigma_n}$, there is a $x^* \in [x_i^*]_{i\in\sigma_n \bigcup \eta_n}$ so that $\|x^*\| = 1$ and $|x^*(x)| > (1 - \frac{1}{n+1})\|x\|$.

Once we have this, by the proof of Theorem IV.4 in [15], we have $[x_i]_{i\in\sigma}^{\perp}$ is the w^* closure of $[x_i^*]_{i\in\eta}$. Put $Y = [x_i^*]_{i\in\eta}^{\perp} = [x_i]_{i\in\sigma}$. By the analogue of Proposition II.1(a) in [15], we deduce that X/Y has an FDD and that $([x_i]_{i\in\sigma_n})_{n=1}^{\infty}$ forms an FDD for Y. So to prove Lemma III.B.10, it is enough to prove that for any blocking (Σ_n) of (σ_n) or any subsequence (σ_{n_k}) of (σ_n) (this of course needs the redefining of (η_n)), (i) and (ii) still hold. But this is more or less obvious because if $\Sigma_n = \bigcup_{i=k_{n-1}+1}^{k_n} \sigma_i$, then we define $\Delta_n = \bigcup_{i=k_{n-1}+1}^{k_n} \eta_i$ and it is easy to check $\{\Sigma_n, \Delta_n\}$ satisfy (i) and (ii). For a subsequence (σ_{n_k}), if we let $\Sigma_k = \sigma_{n_k}$ and define $\Delta_k = \bigcup_{i=n_k}^{n_{k+1}-1} \eta_i$, then $\{\Sigma_n, \Delta_n\}$ satisfy (i) and (ii). The rest is exactly the same as in the proof of Theorem IV.4 in [15].

The next lemma shows that for a reflexive space with an USB FDD, its dual also has an USB FDD.

Lemma III.B.11. *Let X be a reflexive Banach space with an USB FDD (E_n). Then there is a blocking (F_n) of (E_n) so that (F_n^*) is an USB FDD for X^*.*

Proof. Without loss of generality, we assume (E_n) is monotone. Let (δ_i) be a sequence of positive numbers deceasing fast to 0. By the "killing the overlap" technique, we get a blocking (F_n) of (E_n) with $F_n = \sum_{i=k_{n-1}+1}^{k_n}$ so that given any $x = \sum x_i$ with $x_i \in E_i, \|x\| = 1$, there is an increasing sequence (t_n) with $k_{n-1} < t_n < k_n$ such that $\|x_{t_i}\| < \delta_i$, where $0 = k_0 < k_1 < ....$ Let (F_n^*) be the dual FDD of (F_n) and let (x_i^*) be a normalized skipped block sequence with respect to (F_n^*) so that $x_i^* \in \oplus_{j=m_{i-1}+1}^{m_i-1} F_j^*$ where $0 = m_0 < m_1 < ....$ Let $x^* = \sum a_i x_i^*$ with $\|x^*\| = 1$. Let $x = \sum x_i$ be a norming functional of x^* with $x_i \in E_i$. By the definition of (F_n), we get an increasing sequence (t_i) with $k_{i-1} < t_i < k_i$ so that $\|x_{t_i}\| < \delta_i$. We define $y_1 = \sum_{j=1}^{t_{m_1}-1} x_j$ and $y_i = \sum_{j=t_{m_{i-1}}+1}^{t_{m_i}-1} x_j$ for $i > 1$. Let $y = \sum y_i$. So by triangle inequality,

$$\|x - y\| \leq \|\sum x_{m_i}\| \leq \sum \|x_{m_i}\| < \sum \delta_{m_i}. \tag{3.1}$$

Let $(\epsilon_i) \subset \{-1, 1\}^{\mathbf{N}}$ and let $\tilde{x^*} = \sum \epsilon_i a_i x_i^*$. We will estimate $\tilde{x^*}(\sum \epsilon_i y_i)$.

$$|\tilde{x^*}(\sum \epsilon_i y_i)| = |\sum \epsilon_i a_i x_i^*(\sum \epsilon_i y_i)| = |\sum a_i x_i^*(\sum y_i)| = |x^*(y)| \geq 1 - \sum \delta_{m_i}. \tag{3.2}$$

Since (y_i) is a skipped block sequence with respect to (E_i), (y_i) is unconditional. Hence

$$\|\sum \epsilon_i y_i\| \leq C\|\sum y_i\| < C(1 + \sum \delta_{m_i}), \tag{3.3}$$

where C is the unconditional constant associated with the USB FDD (E_n). If we let $\sum \delta_i < \epsilon/2$, then we conclude that

$$\|\tilde{x^*}\| > (1 - \epsilon)/C(1 + \epsilon). \tag{3.4}$$

Therefore, (x_i^*) is unconditional with unconditional constant less that $(1 + 3\epsilon)C$ if ϵ is sufficiently small. Hence (F_n^*) is an USB FDD.

Theorem III.B.12. *Let X be a separable reflexive Banach space. Then the following are equivalent.*

(a) X has the UTP.

(b) X embeds into a reflexive Banach space with an USB FDD.

(c) X^* has the UTP.

Proof. It is obvious that (b) implies (a). If we can prove (a) implies (b), and X satisfies (b), then by Lemma III.B.11, X^* is a quotient of a reflexive space with an USB FDD. So by Theorem III.B.8, X^* has the UTP. Hence we only need to show that (a) implies [b]. Let X_1 be a subspace of X with an FDD (E_n) given by Lemma III.B.10. By Proposition III.B.4 in [26], we get a blocking (F_n) of (E_n) so that (F_n) is an USB FDD. Let $Y_1 = [F_{4n}]$ and $Y_2 = [F_{4n+2}]$. Then (F_{4n}) and (F_{4n+2}) form unconditional FDDs for Y_1 and Y_2. By Lemma III.B.10, X/Y_i has an FDD. Since X has the UTP, by Theorem III.B.8, X/Y_i has the UTP. Now using Proposition III.B.4 in [26] again, we know that X/Y_i has an USB FDD. Noticing that $Y_1 \cap Y_2 = \{0\}$ and $Y_1 + Y_2$ is closed, by Lemma III.B.9, we have that X embeds into $X/Y_1 \oplus X/Y_2$. Hence X embeds into a reflexive space with an USB FDD.

Corollary III.B.13. *Let X be a separable reflexive Banach space with the UTP. Then X embeds into a reflexive Banach space with an unconditional basis.*

Proof. By Theorem III.B.12, X embeds into a reflexive space Y with an USB FDD (E_n). We prove that Y embeds into a reflexive space with an unconditional FDD. Then as was mentioned in the introduction, Y embeds into a reflexive space with an unconditional basis and so X does.

By Lemma III.B.11, there is a blocking (F_n) of (E_n) so that (F_n^*) is an USB FDD for Y^*. Now let $Y_1 = \oplus F_{4n}$ and let $Y_2 = \oplus F_{4n+2}$. Then we have $Y_1 \cap Y_2 = \{0\}$ and $Y_1 + Y_2$ is closed because (F_{2n}), being a skipped block of (E_n), is unconditional. By Lemma III.B.9, Y embeds into $Y/Y_1 \oplus Y/Y_2$. Since $(Y/Y_i)^*$ is isomorphic to $Y_i^\perp$, it is enough to prove $Y_i^\perp$ has an unconditional FDD. Let $G_n^* = F_{4n-3}^* \oplus F_{4n-2}^* \oplus F_{4n-1}^*$.

It is easy to see that (G_n^*) forms an FDD for $Y_1^{\perp}$. Noticing that (G_n) is a skipped block of (F_n^*), we conclude that (G_n) is unconditional. Similarly, we can show that $Y_2^{\perp}$ admits an unconditional FDD. This finishes the proof.

Corollary III.B.14. *Let X be a quotient of a reflexive Banach space with an unconditional FDD. Then X embeds into a reflexive Banach space with an unconditional basis.*

Proof. Combine Theorem III.B.8 and Corollary III.B.13.

We mention again that in 1974 Davis, Figiel, Johnson and Pełczyński proved [4] that a reflexive Banach space X which embeds into a Banach space with a shrinking unconditional basis embeds into a reflexive space X with an unconditional basis. The next year, Figiel, Johnson and Tzafriri [7] got a stronger result by removing the shrinkingness of the unconditional basis in the hypothesis. Our next corollary gives a parallel result for quotients.

Corollary III.B.15. *Let X be a separable reflexive Banach space. If X is a quotient of a Banach space with a shrinking unconditional basis, then X is isomorphic to a quotient of a reflexive Banach space with an unconditional basis.*

Proof. Since X is a quotient of a Banach space with a shrinking unconditional basis, X^* is a subspace of a Banach space with an unconditional basis. Hence, by [7], X^* is isomorphic to a subspace of a reflexive Banach space with an unconditional basis. Therefore, X is isomorphic to a quotient of a reflexive Banach space with an unconditional basis.

Remark III.B.16. Corollary III.B.15 is different from the result of Figiel, Johnson and Tzafriri in that the shrinkingness in our result cannot be removed. The reason is more or less obvious since every separable Banach space is a quotient of ℓ_1 which has an unconditional basis.

Gluing Theorem III.B.12, Corollary III.B.13, Corollary III.B.14 and Corollary III.B.15 together, we have the following long list of equivalences.

Theorem III.B.17. *Let X be a separable reflexive Banach space. Then the following are equivalent.*

(a) X has the UTP.

(b) X is isomorphic to a subspace of a Banach space with an unconditional basis.

(c) X is isomorphic to a subspace of a reflexive space with an unconditional basis.

(d) X is isomorphic to a quotient of a Banach space with a shrinking unconditional basis.

(e) X is isomorphic to a quotient of a reflexive space with an unconditional basis.

(f) X is isomorphic to a subspace of a quotient of a reflexive space with an unconditional basis.

(g) X is isomorphic to a subspace of a reflexive quotient of a Banach space with a shrinking unconditional basis.

(h) X is isomorphic to a quotient of a subspace of a reflexive space with an unconditional basis.

(i) X is isomorphic to a quotient of a reflexive subspace of a Banach space with a shrinking unconditional basis.

C. Example

In this section we give an example of a reflexive Banach space for which there exists a $C > 1$ so that every normalized weakly null sequence admits an C-unconditional subsequence while for any $D > 1$ there is a normalized weakly null tree such that every branch is not D-unconditional. The construction is an analogue of Odell and Schlumprecht's example (see Example 4.2 in [25]).

We will first construct an infinite sequence of reflexive Banach spaces X_n. Each X_n is infinite dimensional and has the property that for $\epsilon > 0$, every normalized weakly null sequence has a $1+\epsilon$-unconditional basic subsequence, while there is a normalized weakly null tree for which every branch is at least C_n-unconditional and C_n goes to infinity when n goes to infinity. Then the ℓ_2 sum X_n is a reflexive Banach space with the desired property.

Let $[\mathbf{N}]^{\leq n}$ be the set of all subsets of the positive integers with cardinality less than or equal to n. Let $c_{00}([\mathbf{N}]^{\leq n})$ be the space of sequences with finite support indexed by $[\mathbf{N}]^{\leq n}$ and denote its canonical basis by $(e_A)_{A\in[\mathbf{N}]^{\leq n}}$. Let (h_i) be any normalized conditional basic sequence which satisfies a block lower l_2 estimate, for example, the boundedly complete basis of James space (see problem 6.41 in [6]). Let $\sum a_A e_A$ be an element of $c_{00}([\mathbf{N}]^{\leq n})$. Let $(\beta_k)_{k=1}^m$ be disjoint segments. By a segment in $[\mathbf{N}]^{\leq n}$, we mean a sequence $(A_i)_{i=1}^k \in [\mathbf{N}]^{\leq n}$ with $A_1 = \{n_1, n_2, ..., n_l\}, A_2 = \{n_1, n_2, ..., n_l, n_{l+1}\}, ..., A_k = \{n_1, n_2, ..., n_l, ..., n_{l+k-1}\}$, for some $n_1 < n_2 < ... <$

n_{l+k-1}. Let $\beta_k = \{A_{1,k}, A_{2,k}, ..., A_{j_k,k}\}$ with $A_{i,k} < A_{i+1,k}$ under the tree order in $[\mathbf{N}]^{\leq n}$. Now we define X_n to be the completion of $c_{00}([\mathbf{N}]^{\leq n})$ under the norm

$$\|\sum a_A e_A\|_{X_n} = \sup\{(\sum_{k=1}^{m}(\|\sum_{A_{i,k}\in\beta_k} a_{A_{i,k}} h_i\|)^2)^{1/2} : (\beta_k)_{k=1}^m \quad \text{are disjoint segments}\}.$$

Let $X = (\sum X_n)_2$. Let C_M be the unconditional constant of $(h_i)_{i=1}^M$. It is clear that C_M tends to infity when M goes to infinity. The normalized weakly null tree $(e_A)_{A\in[\mathbf{N}]^{\leq M}}$ in X_M has the property that every branch of it is 1-equivalent to $(h_i)_{i=1}^M$ since (h_i) has a block lower ℓ_2 estimate with constant 1. So what is remaining is to verify that for every $\epsilon > 0$, every normalized weakly null sequence in X has an $1+\epsilon$-unconditional basic subsequence. Actually, we will prove that there is a subsequence which is $1+\epsilon$-equivalent to the unit vector basis of ℓ_2. By a gliding-hump argument, it is not hard to verify the following fact.

Fact. Let (Y_k) be a sequence of reflexive Banach spaces. And let $Y = (\sum Y_k)_{\ell_2}$. If for every $\epsilon > 0, k \in \mathbf{N}$, every normalized weakly null sequence in Y_k has a subsequence which is $1+\epsilon$-equivalent to the unit vector basis of ℓ_2, then for every $\epsilon > 0$, every normalized weakly null sequence in Y has a subsequence which is $1+\epsilon$-equivalent to the unit vector basis of ℓ_p.

Considering the fact, it is enough to show that for every $\epsilon > 0, k \in \mathbf{N}$, every normalized weakly null sequence in X_k has a subsequence which is $1+\epsilon$ equivalent to the unit vector basis of ℓ_2. We prove this by induction.

For $k = 1$, X_1 is isometric to ℓ_2, so the conclusion is obvious.

Assume the conclusion is true for X_k. By the definition of X_{k+1}, X_{k+1} is isometric to $(\sum(\mathbf{R} \oplus X_k))_{\ell_2}$ (where $\mathbf{R} \oplus X_k$ has some norm so that $\{0\} \oplus X_k$ is isometric to X_k). Hence by hypothesis and the fact we mentioned above, it is easy to see the conclusion is true in X_{k+1}. This finishes the proof.

Remark III.C.1. The proof of the corresponding induction step in Example 4.2 in [25] is more complicated than the very simple induction argument in the previous paragraph. Schlumprecht realized after [25] was written that the induction could be done so simply and his argument works in our context.

CHAPTER IV

OPERATORS WHICH FACTOR THROUGH ℓ_P OR c_0

A. Introduction

In [12], W. B. Johnson answered the following question about the relation between the structure of L_p and ℓ_p.

Question IV.A.1. Give a Banach space condition so that if X is a subspace of L_p $(1 < p < 2)$ which satisfies the condition, then X embeds isomorphically into ℓ_p.

The equivalent dual question would be:

Question IV.A.2. Give a Banach space condition so that if X is a quotient of L_p which satisfies the condition, then X is isomorphic to a quotient of ℓ_p.

For $p > 2$, W. B. Johnson and E. Odell had already proved in [14] that if a subspace X of L_p has no subspace isomorphic to l_2, then X embeds into ℓ_p. For $p < 2$, W. B. Johnson proved that if there exists a $K > 0$ such that every normalized weakly null sequence in X has a subsequence which is K-equivalent to the unit vector basis of ℓ_p, then X is isomorphic to a subspace of ℓ_p. Further W. B. Johnson also gave a complete answer to the dual question in [12]; namely, a quotient of L_p $(2 < p < \infty)$ which is of type p-Banach-Saks is a quotient of ℓ_p. Recall that an operator T from a Banach space X is of type p-Banach-Saks (where $1 < p < \infty$) if there exists a constant λ such that every normalized weakly null sequence in X has a subsequence

(x_n) which satisfies for $n = 1, 2, ...$

$$\| \sum_{i=1}^{n} Tx_i \| \leq \lambda n^{1/p}.$$

X is said to be of type p-Banach-Saks when the identity operator on X is. From the results above, a more general question naturally arises.

Question IV.A.3. Give a necessary and sufficient condition so that if an operator T from L_p to any Banach space Y satisfies the condition, then T factors through ℓ_p.

It was proved in [11] that a bounded linear operator T into L_p $(2 < p < \infty)$ factors through ℓ_p if and only if T is compact when considered as an operator into L_2. This actually answers the Question IV.A.3 for $1 < p < 2$. In [11], W. B. Johnson conjectured that an operator T from L_p $(2 < p < \infty)$ factors through ℓ_p if and only if T is of type p-Banach-Saks. As mentioned above, This conjecture was verified in [12] in the case when T has closed range. Later, W. B. Johnson discovered in [13] a counterexample in the general case for the conjecture, which led him to formulate a conjecture with a stronger condition. That is, an operator T from L_p $(2 < p < \infty)$ factors through ℓ_p if and only if T satisfies Condition IV.A.4 (when X is L_p).

Condition IV.A.4. T is an operator from X so that for every normalized weakly null sequence $(x_n) \subset X$, there is a subsequence (x_{n_k}), such that

$$\| T(\sum a_k x_{n_k}) \| \leq C(\sum | a_k |^p)^{1/p}, \qquad \forall (a_k) \subset \mathbf{R}.$$

In section 2, we use a space constructed by E. Odell and Th. Schlumprecht in [25] to show that for an operator T from L_p $(2 < p < \infty)$, Condition IV.A.4 does not imply that T factors through ℓ_p. E. Odell and Th. Schlumprecht used this space to disprove W. B. Johnson's conjecture that Condition IV.A.5 and reflexivity of X yield that X embeds into an ℓ_p sum of finite dimensional spaces. They also formulated Condition IV.A.6 and proved that Condition IV.A.6 and reflexivity of X do imply that X embeds into an ℓ_p sum of finite dimensional spaces. Here, Condition IV.A.5 and Condition IV.A.6 are defined as following:

Condition IV.A.5. For all $\epsilon > 0$, every normalized weakly null sequence in X admits a subsequence which is $1+\epsilon$ equivalent to the unit vector basis of ℓ_p.

Condition IV.A.6. There is a $C > 1$ such that every normalized weakly null tree in X admits a branch which is C equivalent to the unit vector basis of ℓ_p.

Motivated by Condition IV.A.6, we formulate a stronger condition than Condition IV.A.4, which is an operator version of Condition IV.A.6.

Condition IV.A.7. For every normalized weakly null tree in X, there is a branch (x_k) so that

$$\| T(\sum a_k x_k) \| \leq C(\sum | a_k |^p)^{1/p}, \qquad \forall (a_k) \subset \mathbf{R}.$$

This condition turns out to be the right one for answering Question IV.A.3 when $X = L_p (2 < p < \infty)$.

B. A counterexample

In this section, we construct an operator T from l_2 into $X = (\sum X_n)_p$ (which will be defined below) which satisfies Condition IV.A.4 but does not factor through ℓ_p for $2 < p < \infty$. Since l_2 is isomorphic to a complemented subspace of L_p, we also get an operator from L_p into $X = (\sum X_n)_p$ which satisfies Condition IV.A.4 but does not factor through ℓ_p.

Let $2 < q < p < \infty$, $X = (\sum X_n)_p$ be the space defined in [25], where X_n is the completion of $c_{00}([\mathbf{N}]^{\le n})$ under the norm

$$\| x \|_n = \sup \left\{ (\sum_{i=1}^{m} \| x|_{\beta_i} \|_q^p)^{1/p} : (\beta_i)_1^m \text{ are disjoint segments in} [\mathbf{N}]^{\le n} \right\}.$$

Here $[\mathbf{N}]^{\le n}$ denotes all sets of natural numbers with cardinality less than n. By a segment in $[\mathbf{N}]^{\le n}$, we mean a sequence $(A_i)_{i=1}^k \in [\mathbf{N}]^{\le n}$ with $A_1 = \{n_1, n_2, ..., n_l\}$, $A_2 = \{n_1, n_2, ..., n_l, n_{l+1}\}, ..., A_k = \{n_1, n_2, ..., n_l, ..., n_{l+k-1}\}$, for some $n_1 < n_2 < ... < n_{l+k-1}$. A branch in $[\mathbf{N}]^{\le n}$ is a maximal segment in $[\mathbf{N}]^{\le n}$.

Remark IV.B.1. The node basis $(\tilde{e}_A^n)_{A \in [\mathbf{N}]^{\le n}}$ given by $\tilde{e}_A^n(B) = \delta_{A,B}$ for any $B \in [\mathbf{N}]^{\le n}$ is a 1-unconditional basis for X_n. Moreover, $(\tilde{e}_{A_i}^n)_1^n$ is 1-equivalent to the unit vector basis of ℓ_q^n if $(A_i)_1^n$ is a branch in $[\mathbf{N}]^{\le n}$.

If we write $l_2 = (\sum l_2)_2$, $(e_A^n)_{A \in [\mathbf{N}]^{\le n}}$ is the unit vector basis of the n-th l_2 and $(\tilde{e}_A^n)_{A \in [\mathbf{N}]^{\le n}}$ is the unit vector basis of X_n, then the operator $T : l_2 \to X = (\sum X_n)_p$ is defined so that:

$$T(e_A^n) = \tilde{e}_A^n.$$

Since $2 < q < p$ we can linearly extend T to be an operator of norm one from l_2 into X.

Claim 1. Operator T satisfies Condition IV.A.4.

Let (x_n) be a normalized weakly null sequence in l_2, and $\epsilon > 0$. Then $(T(x_n))$ is a weakly null sequence in $(\sum X_n)_p$. By the proof of Example 4.2 in [25], we can pick a subsequence (x_{n_k}) such that for all $(a_k) \subset \mathbf{N}$

$$\begin{aligned} \| T(\sum a_k x_{n_k}) \| &\leq 2(\sum \|T(a_k x_{n_k})\|^p)^{1/p} \\ &\leq 2(\sum \mid a_k \mid^p)^{1/p}. \end{aligned}$$

So we proved Claim 1. Our second claim is

Claim 2. T does not factor through ℓ_p.

In order to prove the claim, we need the following lemma which is an application of a result concerning blockings of F.D.D.'s proved in [17]. This result was reformulated as Proposition 1.g.4. in [20].

Lemma IV.B.2 *Let $p > 2$, then any bounded linear operator A from l_2 into ℓ_p factors through $(\sum E_n)_{\ell_p}$ in such a way that $A = A' \circ J$, where (E_n) is a blocking of the canonical basis of l_2 and J is the formal identity from l_2 into $(\sum E_n)_{\ell_p}$.*

Proof. By Proposition 1.g.4. in [20], we find a blocking (E_n) of the canonical basis of l_2 such that $A(E_n)$ is essentially contained in $F_{n-1} \bigoplus F_n$, where (F_n) is a blocking of the canonical basis of ℓ_p. Let J be the formal identity map from l_2 into $(\sum E_n)_{\ell_p}$. Since $p > 2$, J is always bounded. Let A' be the linear map from $(\sum E_n)_{\ell_p}$ into ℓ_p such that $A = A' \circ J$. We claim that A' is bounded. Actually, let $x = \sum x_n$ with

$x_n \in E_n$. Then by the construction of (E_n) and (F_n), we have

$$\begin{aligned}\|A'(x)\| &\leq \|A'(\sum x_{2n})\| + \|A'(\sum x_{2n-1})\| \\ &\leq (\|A\| + \epsilon)((\sum \|x_{2n}\|^p)^{1/p} + (\sum \|x_{2n-1}\|^p)^{1/p}) \\ &\leq 2(\|A\| + \epsilon)(\sum \|x_n\|^p)^{1/p}.\end{aligned}$$

So A' is bounded.

Now we can prove Claim 2.

Proof. Suppose T factors through ℓ_p. Then by Lemma IV.B.2, T factors through $(\sum E_n)_{\ell_p}$ for some blocking of the canonical basis of l_2. Let $T = J_1 \circ J_2$, where J_1 is the formal identity from l_2 into $(\sum E_n)_{\ell_p}$ and J_2 is a bounded linear operator from $(\sum E_n)_{\ell_p}$ into $(\sum X_n)_{\ell_p}$. Since T is the formal identity from l_2 into $(\sum X_n)_{\ell_p}$, we deduce that J_2 is also a formal identity. By the choice of (E_n) and the definition of X_n, for any $k \in \mathbf{N}$, we can find a finite basic subsequence $(e^k_{A_n})^k_{n=1}$ of l_2 such that $e^k_{A_n}$'s sit in different E_{r_n}'s and $(A_n)^k_1$ is a branch of $[\mathbf{N}]^{\leq k}$. J_2 is the formal identity, so $J_2(e^k_{A_n}) = \tilde{e}^k_{A_n}$, hence $\|J_2\| \geq k^{1/q-1/p}$. Since k is arbitrary, this shows that J_2 is not bounded. This is a contradiction.

C. Main result

Now we give the sufficient condition for an operator from L_p $(2 < p < \infty)$ to factor through ℓ_p.

Definition IV.C.1. Let $1 \leq p < \infty, C > 0$ and X, Y be Banach spaces. $T : X \to Y$

is a bounded linear operator. We say that T satisfies an upper-(C,p)-tree estimate, if for every normalized weakly null tree in X, there exists a branch (x_i) such that

$$\|T(\sum a_i x_i)\| \leq C(\sum \mid a_i \mid^p)^{1/p}, \qquad \forall (a_i) \subset \mathbf{R}.$$

When $p = \infty$, T satisfies an upper-(C,∞)-tree estimate, if for every normalized weakly null tree in X, there exists a branch (x_i) such that

$$\sup_n\{\|T(\sum_{i=1}^{n} x_i)\|\} \leq C.$$

Theorem IV.C.2. *Let $2 < p < \infty, X$ be a Banach space, $T : L_p \to X$ be a bounded linear operator. Then T satisfies an upper-(C,p)-tree estimate if and only if T factors through ℓ_p.*

As preparation for the proof, we present the following known lemmas (see [12]).

Lemma IV.C.3. *Let $2 < p < \infty$, X be a Banach space, and let $T : L_p \to X$ be a bounded linear operator. Then T factors through ℓ_p if and only if there are a blocking (H_n) of the Haar system and a bounded linear operator $S : (\sum(H_n, \|\cdot\|_p))_{\ell_p} \to X$, such that $T = S \circ J$, where J is the formal identity map from L_p into $(\sum(H_n, \|\cdot\|_p))_{\ell_p}$.*

Remark IV.C.4. Since $2 < p < \infty$, the formal identity map J from L_p into $(\sum(H_n, \|\cdot\|_p))_p$ is always bounded.

Proof. For any blocking (H_n) of the Haar system, since H_n is finite dimensional and uniformly complemented in L_p, it is uniformly complemented in ℓ_p. So

$(\sum(H_n, \|\cdot\|_p))_{\ell_p}$ is complemented in ℓ_p, hence ismorphic to ℓ_p by [28] (or Theorem 2.a.3. in [20]). On the other hand, by Theorem II.1 in [12] any operator T from L_p into ℓ_p factors through $(\sum(H_n, \|\cdot\|_p))_{\ell_p}$ for some blocking (H_n) of the Haar system in the way that $T = S \circ J$ where J is the formal identity.

Lemma IV.C.5. *Let $2 < p < \infty$, X be a Banach space, $T : L_p \to X$ be a bounded linear operator and (H_n) be a blocking of the Haar system. Then there is a bounded linear operator $S : (\sum(H_n, \|\cdot\|_p))_{\ell_p} \to X$, such that $T = S \circ J$, where J is the formal identity map from L_p into $(\sum(H_n, \|\cdot\|_p))_{\ell_p}$, if and only if $\exists C > 0$ s.t.*

$$\| T(\sum a_k x_k) \| \leq C(\sum | a_k |^p)^{1/p}, \qquad \forall (a_k) \subset \mathbf{R}, x_k \in S_{H_k}. \tag{4.1}$$

Proof. Inequality 4.1 is equivalent to saying that the map $Q : J(L_p) \to X$ which satisfies $T = Q \circ J$ is bounded. Considering Remark IV.C.4 and noticing that $J(L_p)$ is obviously dense in $(\sum(H_n, \|\cdot\|_p))_{\ell_p}$, we are done.

Definition IV.C.6. (x_n) is said to be a block sequence w.r.t. (E_n) if there exists a sequence of integers $0 = m_1 < m_2 < m_3 < \ldots$ such that $x_n \in \bigoplus_{j=m_n}^{m_{n+1}-1} E_j, \forall n \in \mathbf{N}$. (x_n) is said to be a skipped-block sequence w.r.t. (E_n) if there exists a sequence of increasing integers $0 = m_1 < m_2 < m_3 < \ldots$ such that $m_n + 1 < m_{n+1}$ and $x_n \in \bigoplus_{j=m_n+1}^{m_{n+1}-1} E_j, \forall n \in \mathbf{N}$. Two skipped-block sequences (x_n) and (y_n) are said to be *intrusive* if $x_1, y_1, x_2, y_2, \ldots$ or $y_1, x_1, y_2, x_2, \ldots$ is a block sequence.

Definition IV.C.7. A property $P(C)$ with some parameter $C > 0$ for normalized block sequences in X is said to be *closed under combination* if there is a $C' > 0$ depending only on C such that for any two *intrusive* normalized block sequences

$(x_n), n \in \mathbf{N}$ and $(y_n), n \in \mathbf{N}$ satisfying $P(C)$, the natural combination sequence $x_1, y_1, x_2, y_2, ...$ or $y_1, x_1, y_2, x_2, ...$ satisfies $P(C')$. For any $C > 0$ and $\epsilon > 0$, if there exists $(\delta_i) \searrow 0$ so that for any normalized sequence (x_n) that has property $P(C)$ with $x_n \in F_n$ for some blocking (F_n) of (E_n), we have that any sequence (y_n) with $y_n \in F_n$ and $\| y_n - x_n \| < \delta_n$ has property $P(C + \epsilon)$, then we say P is *stable under small perturbations*.

Definition IV.C.8. Let $C > 0$. A normalized block sequence (x_n) is said to be C-*good* if (x_n) satisfies property $P(C)$. Otherwise we say that it is C-*bad*. A branch of a normalized block tree is C-*good* if it is a C-*good* sequence. A blocking (F_n) of (E_n) is C-*good* if all normalized sequences (x_n) with $x_n \in F_n$ satisfy property $P(C)$. A blocking (F_n) of (E_n) is C-*semigood* if all normalized sequences (x_n) with $x_n \in F_{2n}$ satisfy property $P(C)$.

Remark IV.C.9. If for every blocking (F_n) of (E_n), (F_n) is C-*semigood*, then we have that any skipped-block sequence (x_n) w.r.t. (E_n) is C-*good*. On the other hand, if any skipped-block sequence w.r.t. (E_n) is C-good, then all blockings of (E_n) are C-*semigood*.

Definition IV.C.10. We say x *sits* in a block of (E_n) if $x = \sum_{i=k_1}^{k_2} x_i$ with $x_i \in E_i$. Let $y = \sum_{i=m_1}^{m_2} y_i$ with $y_i \in E_i$. If $k_2 < m_1$, then we say y *sits farther than* x. A normalized block tree w.r.t. (E_n) is a family $(x_A)_{A \in [\mathbf{N}]^{<\omega}} \subset S_X$ such that

(a) For any $A \in [\mathbf{N}]^{<\omega}$, x_A *sits* in some block of (E_n).

(b) If A is a proper initial segment of B, then x_B *sits farther than* x_A.

(c) If $\max A < n < m$, then $x_{A \bigcup \{m\}}$ *sits farther than* $x_{A \bigcup \{n\}}$.

Proposition IV.C.11. *Let X be a Banach space with an F.D.D (E_n). Consider the three conditions:*

(i) *There exists a $C > 0$ such that every blocking of (E_n) has a further blocking (F_n) so that all further blockings of $(F_n)_{n>1}$ are C-good.*

(ii) *There exists a $C > 0$ such that every blocking of (E_n) has a further blocking (F_n) so that all further blockings of (F_n) are C-semigood.*

(iii) *There exists a $C > 0$ such that every normalized block tree w.r.t (E_n) in X has a C-good branch.*

Then we have:

(a) *(i) implies (ii) and (ii) implies (iii).*

(b) *If property P is closed under combination, then (ii) implies (i).*

(c) *If property P is stable under small perturbations and makes $D_C^{'}$ closed under the pointwise topology on $[\mathbf{N}]^\omega, \forall C > 0$, then (iii) implies (ii).*

where $D_C^{'}$ is defined as:

$$D_C^{'} = \{M \in [\mathbf{N}]^\omega : \text{the blocking of} \quad E_n \quad \text{corresponding to} \quad M \quad \text{is } C\text{-}\textit{semigood}\}.$$

$[\mathbf{N}]^\omega$ *denotes the set of all infinite subsets of positive integers. For a blocking (F_n) of (E_n), given by $F_n = \sum_{i=n_{i-1}+1}^{n_i} E_i$ and $n_0 = 0$, we say that (F_n) corresponds to the set $\{n_1, n_2, ...\}$.*

Proof. Since (a) and (b) trivially follow from the definitions above, we omit the proof.

It remains to prove that (iii) implies (ii) when D_C' is closed under pointwise topology on $[\mathbf{N}]^\omega$. This is essentially contained in Theorem 3.3 in [25]. For the convenience of the reader, we write down a direct argument which includes only the part of the proof of Theorem 3.3 in [25] that is needed. For any $C > 0$, set

$$D_C = \{\text{blockings of} \quad (E_n) \quad \text{which are } C\text{-}semigood\}.$$

So we can identify D_C with

$$D_C' = \{M \in [\mathbf{N}]^\omega : \text{the blocking corresponding to} \quad M \quad \text{is } C\text{-}semigood\}.$$

Let (G_n) be any blocking of (E_n). Since property P makes D_C' closed under the pointwise topology on $[\mathbf{N}]^\omega$, then by the infinite version of Ramsey's theorem (cf. [23]), there are two cases:

1. there is a blocking (F_n) of (G_n) all further blockings of which are C-*semigood.*

2. there is a blocking (F_n) of (G_n) any further blocking of which is not C-*semigood.*

In the first case, we are done. In the second case, we will construct a block tree which results in a contradiction. Let N' be the infinite subset of positive integers corresponding to the blocking (F_n) any further blocking of which is not C-*semigood.* Then for each $\tilde{M} \in [N']^\omega$ (which corresponds to a further blocking of (F_n)), we can pick a C-*bad* sequence $(x_i^{\tilde{M}})$ which is a skipped blocked sequence relative to the blocking corresponding to $\tilde{M}$. Letting $N' = \{n_1, n_2, n_3, ...\}$, we know that for any $\tilde{M} \in [\{n_3, n_4, ...\}]^\omega$

$$x_1^{\{n_1,n_2\} \bigcup \tilde{M}} \in S_{[E_i]|_{n_1+1}^{n_2}}.$$

By Ramsey's theorem and the compactness of $S_{[E_i]|_1^{n_2}}$, we can find an $x_{\{1\}} \in S_{[E_i]|_{n_1+1}^{n_2}}$

and an $\tilde{M}^1 \subset \{n_3, n_4, ...\}$ such that for all $\tilde{M} \in [\tilde{M}^1]^\omega$, we have

$$\| x_{\{1\}} - x_1^{\{n_1,n_2\}\bigcup \tilde{M}} \|< \delta_1.$$

Doing the procedure again, we can find an $x_{\{2\}} \in S_{[E_i]|^{n_2'}_{1+n_1'}}$ and an $\tilde{M}^2 \in [\tilde{M}^1]^\omega$ so that for all $\tilde{M} \in [\tilde{M}^2]^\omega$, we have

$$\| x_{\{2\}} - x_1^{\{n_1',n_2'\}\bigcup \tilde{M}} \|< \delta_1,$$

where n_1', n_2' are the first two elements of $\tilde{M}^2$. Continuing this procedure, we get x_i for all $i \in \mathbf{N}$. For the second level of the tree, by using the same method as above, we can find for x_1 an $x_{1,2} \in S_{[E_i]|^{n_2'}_{1+n_1'}}$ and an $\tilde{M}^{1,2} \in [\tilde{M}^1 - \{n_1', n_2'\}]^\omega$ such that for all $\tilde{M} \in [\tilde{M}^{1,2}]^\omega$, we have

$$\| x_{\{1,2\}} - x_2^{\{n_1,n_2,n_1',n_2'\}\bigcup \tilde{M}} \|< \delta_2.$$

Let $\tilde{n_1^2}, \tilde{n_2^2}$ be the smallest two elements of $\tilde{M}^{1,2}$, we can find our desired $x_{1,3}$ and so on. Since P is *stable under small perturbations*, by continuing this process, we get a normalized block tree w.r.t. (F_n) every branch of which doesn't have property $P(C + \epsilon)$. Since C is arbitrary, we get a contradiction.

Now we can prove our main result.

Proof of Theorem IV.C.2. Given an operator T, we say that a normalized block sequence (x_n) w.r.t. the canonical Haar system (h_n) satisfies property $P(C)$ if

$$\| T(\sum a_i x_i) \|\leq C(\sum \mid a_i \mid^p)^{1/p}, \qquad \forall (a_i) \subset \mathbf{R}.$$

Let (x_n) and (y_n) be two intrusive normalized skipped-block sequences w.r.t. (h_n).

If both of them satisfy property $P(C)$, then

$$\begin{aligned}\| T(\sum(a_ix_i + b_iy_i)) \| &\leq \| T(\sum a_ix_i) \| + \| T(\sum b_iy_i) \| \\ &\leq C((\sum \mid a_i \mid^p)^{1/p} + (\sum \mid b_i \mid^p)^{1/p}) \\ &\leq 2C(\sum(\mid a_i \mid^p + \mid b_i \mid^p))^{1/p}.\end{aligned}$$

So P is *closed under combination.* Let (H_n) be a blocking of (h_n) and (x_n) be a normalized block sequence with $x_n \in H_n$ which satisfies property $P(C)$. Let (y_n) be another normalized block sequence with $y_n \in H_n$ such that $\| x_n - y_n \| < \delta_n$ where $\delta_n < \epsilon/2^n\|T\|$. Then

$$\begin{aligned}\| T(\sum a_iy_i) \| &\leq \| T(\sum a_ix_i) \| + \| T(\sum a_i(x_i - y_i)) \| \\ &\leq (C + \epsilon)(\sum \mid a_i \mid^p)^{1/p}.\end{aligned}$$

So P is *stable under small perturbations.* Also notice that the set

$$\Omega(C) = \{(x_k) \in S^{\omega}_{L_p} : \| T(\sum a_kx_k) \| \leq C(\sum \mid a_k \mid^p)^{1/p}\}, \qquad \forall (a_k) \subset \mathbf{R}$$

is closed under pointwise limits where $S^{\omega}_{L_p}$ denotes the set of all infinite sequences in the unit sphere of L_p. Then the set

$$D^{'}_C = \{M \in [\mathbf{N}]^{\omega} : \text{the blocking corresponding to} \quad M \quad \text{is } C\textit{-semigood}\}$$

is closed under pointwise limits in $[\mathbf{N}]^{\omega}$. For L_p, since every block tree is a weakly null tree, by hypothesis every block tree has a good branch. So by Proposition IV.C.11 and our argument above, we know that there is a blocking (H_n) of (h_n) and $D < \infty$ such that all block sequences of $(H_n)_{n>1}$ are in $\Omega(D)$. Then it is easy to see that there is a $C' > 0$ so that all block sequences of (H_n) are in $\Omega(C')$. Combining Lemma IV.C.3 and Lemma IV.C.5, we conclude that T factors through ℓ_p.

Remark IV.C.12. If T factors through ℓ_p, say $T = T_1 \circ T_2$ where T_2 is the operator from L_p into ℓ_p and T_1 is the operator from ℓ_p into X, then for any normalized weakly null tree (x_A) in L_p, $(T_2(x_A))$ is a weakly null tree in ℓ_p. Hence there is a branch of (x_A) which satisfies an upper-(2,p)-tree estimate. So the upper-(C,p)-tree estimate is also a necessary condition.

Actually we have the following generalization of Theorem IV.C.2.

Definition IV.C.13. Let $1 \leq p \leq \infty$. Let X be a Banach space with an F.D.D. (E_n). We say (E_n) satisfies a block lower-p estimate if there exists a $C > 0$ such that for any block basis (x_n) with respect to (E_n),

$$\| \sum x_n \| \geq C (\sum \|x_n\|^p)^{1/p}.$$

Theorem IV.C.14. *Let $1 < p \leq \infty$ and X be a Banch space with a shrinking F.D.D. (E_n) which satisfies a block lower-p estimate. Let $T : X \to Y$ be a bounded linear operator which satisfies an upper-(C,p)-tree estimate. If $p < \infty$, then T factors through $(\sum F_n)_{\ell_p}$ and if $p = \infty$, T factors through $(\sum F_n)_{c_0}$ for some blocking (F_n) of (E_n).*

Proof. Let $p < \infty$. Let (F_n) be any blocking of (E_n) and $J_F : (\sum F_n)_Z \to (\sum F_n)_{\ell_p}$ be the formal identity map. Since (E_n) satisfies a block lower-p estimate, J_F is always bounded. If the map $S_F : J_F(X) \to Y$ with $T = S_F \circ J_F|_X$ is bounded, i.e. there exists a $C > 0$ such that for all (x_k) with $x_k \in F_k$ and $j \in \mathbf{N}$,

$$\| T(\sum_{k=1}^{j} a_k x_k) \| \leq C (\sum | a_k |^p)^{1/p} \qquad \forall (a_k) \subset \mathbf{R},$$

then T factors through the subspace $J_F[X]$ of $(\sum F_n)_{\ell_p}$. Since $J_F[X]$ is dense in $(\sum F_n)_{\ell_p}$, the operator S_F can be extended to the whole space $(\sum F_n)_{\ell_p}$. Hence T factors through $(\sum F_n)_{\ell_p}$. For an operator T, we say that a normalized block sequence (x_n) w.r.t. (E_n) satisfies property $P(C)$ if for all $j \in \mathbf{N}$,

$$\| T(\sum_{i=1}^{j} a_i x_i) \| \leq C(\sum | a_i |^p)^{1/p}, \qquad \forall (a_i) \subset \mathbf{R}.$$

As in the proof of Theorem IV.C.2, we can check that property P is closed under combination and stable under small perturbation. Since (E_n) is shrinking, every block tree is weakly null, hence by hypothesis every block tree has a good branch. Now by applying Proposition IV.C.11, we have that there is a blocking (F_n) of (E_n) so that the operator S_F defined above is bounded. The proof above works as well when $p = \infty$.

A further question is what if X is only a subspace of a space with a shrinking F.D.D. In the case when p is finite, we can prove the following generalization of Theorem IV.C.2 by using the method in the proof of Theorem 4.1 in [25].

Theorem IV.C.15. *Let $1 < p < \infty$ and X be a subspace of a space Z with a shrinking F.D.D. (E_n) which satisfies a block lower-p estimate. Let $T : X \to Y$ be a bounded linear operator which satisfies an upper-(C,p)-tree estimate. Then T factors through a subspace of $(\sum F_n)_{\ell_p}$, where (F_n) is a blocking of (E_n).*

In order to prove the above theorem, we need Lemma IV.C.16, which is a result of W. B. Johnson restated as Corollary 4.4 in [25].

Lemma IV.C.16. *Let X be a subspace of the reflexive space Z and let (F_i) be* *an F.D.D. for Z. Let $\delta_i \downarrow 0$. There exists a blocking (G_i) of (F_i) given by $G_i = \bigoplus_{j=N_{i-1}+1}^{N_i} F_j$ for some $0 = N_0 < N_1 < \ldots$ with the following property. For all $x \in S_X$ there exists $(x_i)_1^\infty \subset X$ and $t_i \in (N_{i-1}, N_i]$ for $i \in \mathbf{N}$ so that:*

a) $x = \sum_{i=1}^\infty x_i$.

b) *For $i \in \mathbf{N}$, either $\|x_i\| < \delta_i$ or $\|P_{\bigoplus_{j=t_{i-1}+1}^{t_i-1} F_j}(x_i) - x_i\| < \delta_i \|x_i\|$.*

c) *For $i \in \mathbf{N}$, $\|P_{\bigoplus_{j=t_{i-1}+1}^{t_i-1} F_j}(x) - x_i\| < \delta_i$.*

Proof of Theorem IV.C.15. Let (F_n) be any blocking of (E_n) and $J_F : (\sum F_n)_Z \to (\sum F_n)_{\ell_p}$ be the formal identity map. Since (E_n) satisfies a block lower-p estimate, J_F is always bounded. If the map $S_F : J_F(X) \to Y$ with $T = S_F \circ J_F|_X$ is bounded, i.e. there exists a $C > 0$ such that for all $x = \sum a_k x_k \in X$ with $x_k \in S_{F_k}$,

$$\| T(\sum a_k x_k) \| \leq C(\sum \mid a_k \mid^p)^{1/p},$$

then T factors through a subspace of $(\sum F_n)_{\ell_p}$. Let $C > 0$ and set

$$\mathcal{A} = \{(x_i) \in S_X^\omega : \forall j \in \mathbf{N}, \|T(\sum_{i=1}^j a_i x_i)\| \leq C(\sum \mid a_i \mid^p)^{1/p}, \forall (a_i) \subset \mathbf{R}\}.$$

Applying Proposition 2.4 in [26] to the set $\mathcal{A}$, we get a blocking (F_i) of (E_i) such that there exists $\delta = (\delta_i)$ so that if $(x_n) \subset S_X$ is a δ-skipped block w.r.t. (F_n), then whenever $\sum a_i x_i$ converges, we have $\|T(\sum a_i x_i)\| \leq 2C(\sum \mid a_i \mid^p)^{1/p}\}$. Because the F.D.D. (E_i) is shrinking and satisfies a block lower-p estimate, Z is reflexive. Now Let (G_i) be the blocking of (F_i) given by Lemma IV.C.16. Let $x \in S_X, x = \sum x_i = \sum \tilde{x}_i$ with $\tilde{x}_i \in G_i$ and x_i as in Lemma IV.C.16. Let $y_i = P_{\bigoplus_{j=t_{i-1}+1}^{t_i-1} F_j} x$; then there exist C_1, C_2 such that

$$C_1 \max(\|y_i\|, \|y_{i+1}\|) - \delta_i \leq \|\tilde{x}_i\| \leq C_2 \|y_i\| + \delta_i.$$

So when δ_i's are sufficiently small, we have

$$\begin{aligned}\|T(\sum \tilde{x}_i)\| &= \|T(\sum x_i)\| \\ &\leq C(\sum \|x_i\|^p)^{1/p} \\ &\leq 2C(\sum \|y_i\|^p)^{1/p} \\ &\leq C'(\sum \|\tilde{x}_i\|^p)^{1/p}.\end{aligned}$$

This is exactly what we want.

In particular, when Z is $L_p (2 < p < \infty)$, we have the corollary below.

Corollary IV.C.17. *Let $2 < p < \infty$ and let X be a subspace of L_p. If $T : X \to Y$ is a bounded linear operator which satisfies an upper-(C,p)-tree estimate, then T factors through a subspace of ℓ_p.*

For the case when $p = \infty$, we have the following result, the proof of which was shown to me by W. B. Johnson.

Theorem IV.C.18. *Let X be a Banach space with X^* separable. Let $T : X \to Y$ be a bounded linear operator satisfying an upper-(C,∞)-tree estimate. Then T factors through a subspace of c_0.*

To prove the theorem, we need the following Lemma, which is a corollary of Theorem IV.C.14.

Lemma IV.C.19. *Let X be a Banach space with a shrinking F.D.D. (E_i) and let*

$T : X \to Y$ be a bounded linear operator satisfying an upper-(C,∞)-tree estimate. Then T factors through a subspace of c_0.

Proof. By Theorem IV.C.14, we know that T factors through $(\sum F_i)_{c_0}$ for some blocking (F_i) of (E_i). Since $(\sum F_i)_{c_0}$ embeds into c_0, T factors through a subspace of c_0.

Proof of Theorem IV.C.18. For our convenience, without loss of generality, we assume Y is l_∞. Since X^* is separable, by Theorem IV.4 in [15](or see Theorem 1.g.2 in [20]), there is a closed subspace E of X so that both E and X/E have a shrinking F.D.D. Let T_E be the restriction of the operator T to E. By Lemma IV.C.19, T_E factors through a subspace of c_0. We denote $T_E = B \circ A$ where A is the operator from E into c_0 and B is the operator from $A[E]$ into l_∞. Since X is separable and $A[E]$ is in c_0, we can extend A to be defined on X. Let $\tilde{A}$ be the extension. Since $Y = l_\infty$, we can also extend B to be defined on c_0. Let $\tilde{B}$ be the extension. So we get a new operator $\tilde{T} = \tilde{B} \circ \tilde{A}$ which factors through a subspace of c_0 (actually factors through c_0). Now we consider the operator $T - \tilde{T}$. This operator is identically zero on E which also satisfies an upper-(C_1,∞)-tree estimate. So it naturally induces an operator S from X/E into l_∞ ($S(x + E) = (T - \tilde{T})(x)$). If we can prove that S satisfies an upper-(C,∞)-tree estimate, then by Lemma IV.C.19, S factors through a subspace of c_0. Hence $T - \tilde{T}$ factors through a subspace of c_0. Since $\tilde{T}$ factors through a subspace of c_0, we conclude that $T = (T - \tilde{T}) + \tilde{T}$ factors through a subspace of c_0. So it is enough to show S satisfies an upper-(C,∞)-tree estimate. Let us first prove that for any normalized weakly null sequence (z_i) in X/E, there is a subsequence (z_{k_i}) whose pull back (under the canonical quotient $Q : X \to X/E$) (x_i) in X is also weakly null and $\max\{\|x_i\|\} < 2$. Pick a sequence (x_i) in X such that $Q(x_i) = z_i$ and

$\max\{\|x_i\|\} < 1+\epsilon$. Since l_1 does not embed into X, by Rosenthal's l_1 theorem (see [30]) and passing to a subsequence, we can assume (x_i) is weakly cauchy. Since (z_i) is weakly null, we can find convex combinations $y_i = \sum_{j=N_{i-1}+1}^{N_i} \alpha_j z_j$ such that $\|y_i\| < 1/2^i$. Replacing x_i by $x_i - \sum_{j=N_{i-1}+1}^{N_i} \alpha_j x_j$, we get that (x_i) is weakly null and $\|Q(x_i) - z_i\| < 1/2^i$. By replacing x_i by an element in the ball centered at x_i with radius $1/2^i$, we get a weakly null sequence (x_i) such that $Q(x_i) = z_i$ and $\|x_i\| < 2$. For any normalized weakly null tree in X/E, using the result above, it is easy to get by induction that there is a subtree whose pull back in X is also a weakly null tree and the norms of each element of the tree are uniformly bounded. Since $T - \tilde{T}$ satisfies an upper-(C_1,∞)-tree estimate, we conclude that S satisfies an upper-(C,∞)-tree estimate. We are done.

When T is the identity map, in virture of Lemma IV.C.21, we have the following corollary which is a result of Kalton (Theorem 3.2. in [19]).

Corollary IV.C.20. *Let X be a separable Banach space and does not contain l_1. If for every normalized weakly null tree in X, there is a branch (x_i) so that*

$$\sup_n\{\|\sum_{i=1}^{n} x_i\|_X\} \leq C,$$

then X embeds into c_0.

Lemma IV.C.21. *Let $1 < p \leq \infty$. When X is a Banach space with an upper-(C,p)-tree estimate, then the condition - "X is separable and l_1 does not embed into X" and the condition - "X^* is separable" are equivalent".*

Proof of Lemma IV.C.21.

Fact 1. (see Theorem 4.2 in [1]) If l_1 does not embed into X, then $\eta(X) = I_w^+(X)$.

Here $\eta(X)$ is the Szlenk index (see Definition 4.1 in [1]) and $I_w^+(X)$ is the l_1^+-weakly null index (see Definition 3.6 in [1]).

Fact 2. The upper-(C,p)-tree estimate implies that $I_w^+(X) = \omega$.

Fact 3. (see (ix) in Theorem 3.14 in [1]) If l_1 does not embed into X, then $\eta(X) < \omega_1$ is equivalent to X^* is separable.

From the above facts, we know that if l_1 does not embed into X and X satisfies an upper-(C,p)-tree estimate for some $p > 1$, then X^* is separable. The other direction is trivial. So we are done.

CHAPTER V

OPERATORS FROM SEPARABLE REFLEXIVE SPACES WITH ASYMPTOTIC STRUCTURES

A. Introduction

In [26], E. Odell and Th. Schlumprecht prove that a separable reflexive Banach space X which satisfies an (ℓ_p, ℓ_q)-tree estimate embeds into a reflexive Banach space with an (ℓ_p, ℓ_q) FDD. In particular, this proves that if every normalized weakly null tree in a separable reflexive Banach space X has a branch equivalent to the unit vector basis of ℓ_p $(1 < p < \infty)$, then X is isomorphic to a subspace of an ℓ_p sum of finite dimensional spaces. Then, in [27], E. Odell, Th. Schlumprecht and A. Zsak prove that a separable reflexive Banach space X which satisfies an asymptotic (ℓ_p, ℓ_q)-tree estimate embeds into a reflexive Banach space with an asymptotic (ℓ_p, ℓ_q) FDD. So a special case is that every separable reflexive asymptotic ℓ_p space is a subspace of a reflexive Banach space with an asymptotic ℓ_p FDD. Recall that a Banach space X with an FDD (E_n) is asymptotic ℓ_p with respect to (E_n) [22] if there exists a $C > 0$ so that for all n and every normalized block sequence $(x_i)_{i=1}^n$ of $(E_i)_{i=n}^\infty$ is C-equivalent to the unit vector basis of ℓ_p. A coordinate-free version of this notion is in [21]. Let X be an arbitrary Banach space. X is said to be asymptotic ℓ_p if there exists $0 < C < \infty$ so that $\forall n \in \mathbf{N}, \exists Y_1 \in \mathrm{cof}(X)\ \forall y_1 \in S_{Y_1}, ..., \exists Y_n \in \mathrm{cof}(X)$ $\forall y_n \in S_{Y_n}$, $(y_i)_{i=1}^n$ is C-equivalent to the unit vector basis of ℓ_p^n. An FDD $(E_n)_{i=1}^\infty$ is asymptotic (ℓ_p, ℓ_q) if there exists $0 < C < \infty$ such that for all $n \in \mathbf{N}$ and all block sequences $(x_i)_{i=1}^n$ of $(E_n)_{i=n}^\infty$,

$$C^{-1}(\sum_{i=1}^n \|x_i\|^p)^{1/p} \leq \|\sum_{i=1}^n x_i\| \leq C(\sum_{i=1}^n \|x_i\|^q)^{1/q}.$$

If $p = q$, then we say that (E_n) is asymptotic ℓ_p. The results above can be restated in the way that under some conditions, the identity operator on a separable reflexive Banach space factors through a subspace of ℓ_p sum of finite dimensional spaces and under some other conditions, it factors through a subspace of a space with an asymptotic ℓ_p FDD. From this point of view, it is natural to consider general operators from a separable reflexive Banach space. The goal is to find the right conditions under which the operators factor through a subspace of ℓ_p sum of finite dimensional spaces or factor through a subspace of a space with an asymptotic ℓ_p FDD. In Chapter IV, the author proves that if X is a Banach space with an FDD satisfying a block lower-p estimate and T is an operator from X which satisfies an upper-ℓ_p-tree estimate, then T factors through $(F_n)_{\ell_p}$, where (F_n) is a sequence of finite dimensional spaces. An important consequence is that any bounded linear operator from L_p $(2 < p < \infty)$ which satisfies an upper-ℓ_p-tree estimate factors through ℓ_p (actually this is also a necessary condition). In this chapter, the author considers operators from separable reflexive spaces with certain asymptotic structures and similar results are obtained.

B. Definitions and notations

In [26], the notion of lower-p-tree estimate and upper-q-tree estimate is introduced. The following definition is a generalization of it which can be found in [27].

Definition V.B.1. Let V be a Banach space with a 1-unconditional and normalized basis (v_i). We say that a Banach space X satisfies an lower-V-tree estimate if there exists a $C > 0$ such that every normalized weakly null tree has a branch (x_i) so that for all $(a_i) \subset \mathbf{R}$,

$$\|\sum a_i x_i\| \geq C^{-1}\|\sum a_i v_i\|.$$

In this chapter, we need similar conditions for operators which is a further generalization of Odell and Schlumprecht's original notion.

Definition V.B.2. Let U be a Banach space with a 1-unconditional and normalized basis (u_i). Let T be a bounded linear operator from X. We say that T satisfies an upper-U-tree estimate if there exists a $C > 0$ such that every normalized weakly null tree has a branch (x_i) so that for all $(a_i) \subset \mathbf{R}$,

$$\|T(\sum a_i x_i)\| \leq C\|\sum a_i u_i\|.$$

The following definitions came out from the study of spaces with certain asymptotic structures.

Definition V.B.3. Let $0 \leq p \leq \infty$. A reflexive Banach space X satisfies an asymptotic lower-ℓ_p-tree estimate if there exists a $0 < C < \infty$ so that for every $k \in \mathbf{N}$, every normalized weakly null tree of length k in X has a branch $(x_i)_{i=1}^k$ such that for all $(a_i) \subset \mathbf{R}$,

$$\|\sum_{i=1}^k a_i x_i\| \geq C^{-1}(\sum_{i=1}^k \|a_i\|^p)^{1/p}.$$

Definition V.B.4. Let $0 \leq q \leq \infty$ and let T be a bounded linear operator from a Banach space X. T satisfies an asymptotic upper-ℓ_q-tree estimate if there exists a $0 < C < \infty$ so that for every $k \in \mathbf{N}$, every normalized weakly null tree of length k in X has a branch $(x_i)_{i=1}^k$ such that

$$\|\sum_{i=1}^k x_i\| \leq C(\sum_{i=1}^k \|x_i\|^q)^{1/q}.$$

C. Main results

Let (u_i) be a sequence in Banach space U and let (v_i) be a sequence in Banach space V. We say that (u_i) C-dominates (v_i) or (v_i) is C^{-1}-dominated by (u_i) if for all $(a_i) \subset \mathbf{R}$,

$$\|\sum a_i u_i\|_U \geq C\|\sum a_i v_i\|_V.$$

Let (w_i) be an 1-unconditional normalized basis for W. Let $F = (F_n)$ be an FDD for a Banach space Z. Then we define space $Z_W(F)$ to be the completion of $c_{00}(\bigoplus F_n)$ under the norm

$$\|(x_i)\| = \|\sum_j \|x_i\| w_j\|.$$

Theorem V.C.1. *Let U and V be two Banach spaces with 1-unconditional normalized bases (u_i) and (v_i). Let (w_i) be a normalized $C^{'}$-subsymmetric basis for W. Suppose that (u_i) A_1-dominates (w_i) and (w_i) A_2-dominates (v_i). Let X be a separable reflexive Banach space with an FDD (E_n) which satisfies a lower-U-tree estimate. Let T be a bounded linear operator from X into Y which satisfies an upper-V-tree estimate. Then T factors through $X_W(F)$, where $F = (F_n)$ is a blocking of (E_n).*

In the proof of Theorem V.C.1, the following lemmas are used.

Lemma V.C.2. (Proposition 2.4 and Proposition 2.5 in [26]) *Let X be a separable reflexive Banach space which embeds into a reflexive Banach space with an FDD (E_n). Then for $\mathcal{A} \subset S_X^\omega$, the following are equivalent:*

a) For all $\epsilon > 0$, every weakly null tree in S_X has a branch in $\tilde{\mathcal{A}}_\epsilon$.

b) For all $\epsilon > 0$, there exists a blocking (F_n) of (E_n) and $\delta = (\delta_i), \delta_i \downarrow 0$ so that if

$(x_n) \subset S_X$ is a δ-skipped block w.r.t. (F_i) then $(x_n) \in \tilde{\mathcal{A}}_\epsilon$.

Definitions of δ-skipped block sequences, S_X^ω and $\tilde{\mathcal{A}}_\epsilon$ can be found in Definition 2.2 and Definition 2.3 in [26].

Proof of Theorem V.C.1. For a blocking $F = (F_i)$ of $E = (E_n)$, let J_F be the natural embedding of X into $X_W(F)$ so that if $x = \sum x_i$ with $x_i \in F_i$, then $J_F(x) = \sum x_i \in X_W(F)$. We define $\tilde{T}$ to be the operator from $J_F(X)$ into Y so that $\tilde{T} \circ J_F = T$. J_F and $\tilde{T}$ are initially only defined on the linear span of the FDD's. Once they are bounded, they have bounded linear extensions to the closures. So our goal is to find an appropriate blocking F of E so that J_F and $\tilde{T}$ are bounded. Let C be the constant associated with the upper-V-tree estimate for the operator T and set

$$\mathcal{A} = \{(x_i) \in S_X^\omega : \forall j \in \mathbf{N}, \|T(\sum_{i=1}^{j} a_i x_i)\| \leq C\|\sum_{i=1}^{j} a_i v_i\|, \forall (a_i) \subset \mathbf{R}\}.$$

Since T satisfies an upper-V-tree estimate, applying Lemma V.C.2 to the set $\mathcal{A}$, we get a blocking (G_i) of (E_i) such that there exists $\delta = (\delta_i)$ so that if $(x_n) \subset S_X$ is a δ-skipped block sequence with respect to (G_n), then whenever $\sum a_i x_i$ converges, we have $\|T(\sum a_i x_i)\| \leq (1+\epsilon)C\|\sum a_i v_i\|$, where ϵ is any given positive number. Let $\tilde{C}$ be the constant associated with the lower-u-tree estimate for X and set

$$\mathcal{B} = \{(x_i) \in S_X^\omega : \forall j \in \mathbf{N}, \|\sum_{i=1}^{j} a_i x_i\| \geq \tilde{C}\|\sum_{i=1}^{j} a_i u_i\|, \forall (a_i) \subset \mathbf{R}\}.$$

Since X satisfies a lower-U-tree estimate, applying Lemma V.C.2 again to the set $\mathcal{B}$ and properly shrinking δ, we get a blocking (H_i) of (G_i) such that if $(x_n) \subset S_X$ is a δ-skipped block sequence with respect to (H_n), then whenever $\sum a_i x_i$ converges, we have $\|\sum a_i x_i\| \geq (\tilde{C}/(1+\epsilon))\|\sum a_i u_i\|$. From the above arguments, we get a blocking (H_i) of (E_i) so that any δ-skipped block sequence with respect to (H_i) in X $\tilde{C}/(1+\epsilon)$-dominates (u_i) while the image of any δ-skipped block sequence with respect to (H_i) in X under T is $(1+\epsilon)C$-dominated by (v_i). Let $K = \sup_{m<n} \|P_n - P_m\|$ be the projection constant for (E_i), where P_n is the canonical projection from X onto $\oplus_{i=1}^n E_i$. Using the "killing the overlap technique" [12], we can find a further blocking $F = (F_n)$ of (H_n) with $F_n = \oplus_{j=l(n)+1}^{l(n+1)} H_j$, so that for any $x = \sum x_j \in S_X, x_j \in H_j$, there are t_n's with $l(n) < t_n < l(n+1)$ such that $\|x_{t_j}\| < \delta_i$, where $0 = l(1) < l(1) < \ldots.$

First, we prove $\tilde{T}$ is bounded. Let $\epsilon > 0$ be small enough and let $\sum \delta_i < \epsilon$. Denote $x_0 = x_1$. Without loss of generality, we assume $\|T\| = 1$. Let $x = \sum x_i = \sum \tilde{x_j} \in S_X$ with $x_i \in F_i$ and $\tilde{x_j} \in H_j$.

$$
\begin{aligned}
\|\tilde{T}(x)\| &= \|T(\sum x_i)\| \\
&\leq \|T(\sum x_{2i})\| + \|T(\sum x_{2i-1})\| \\
&\leq (1+\epsilon)C(\|\sum \|x_{2i}\|v_i\| + \|\sum \|x_{2i-1}\|v_i\|) \\
&\leq (1+\epsilon)CA_2^{-1}(\|\sum \|x_{2i}\|w_i\| + \|\sum \|x_{2i-1}\|w_i\|) \\
&\leq (1+\epsilon)CC' A_2^{-1}(\|\sum \|x_{2i}\|w_{2i}\| + \|\sum \|x_{2i-1}\|w_{2i-1}\|) \\
&\leq (1+\epsilon)CC'^2 A_2^{-1}(\|\sum \|x_i\|w_i\| + \|\sum \|x_i\|w_i\|) \\
&= 2(1+\epsilon)CC'^2 A_2^{-1}\|x\|_{X_{W(F)}}.
\end{aligned}
$$

Hence $\tilde{T}$ is bounded. What is remaining is to prove that J_F is bounded. Let $t_0 = 1$ and let $y_i = \sum_{j=t_{i-1}}^{t_i - 1} \tilde{x_j}$. Denote $\tilde{y_1} = y_1$ and $\tilde{y_i} = y_i - \tilde{x_{t_{i-1}}}$ for $i \geq 2$. Then we have

$$
\begin{aligned}
\|x\|_{X_{W(F)}} &= \|\sum \|x_i\|w_i\| \\
&\leq A_1^{-1}\|\sum \|x_i\|u_i\| \\
&\leq A_1^{-1}\|\sum 2K(\|y_i\| + \|y_{i+1}\|)u_i\| \\
&\leq 2KA_1^{-1}(\|\sum \|y_i\|u_i\| + \|\sum \|y_{i+1}\|u_i\|) \\
&\leq 2KA_1^{-1}(\|\sum \|\tilde{y_i}\|u_i\| + \|\sum \|\tilde{y_{i+1}}\|u_i\| + 2\epsilon)
\end{aligned}
$$

$$\leq \ 2(1+\epsilon)KA_1^{-1}\tilde{C}^{-1}(\|\sum \tilde{y_i}\| + \|\sum \tilde{y_{i+1}}\| + 2\tilde{C}\epsilon)$$
$$\leq \ 2(1+\epsilon)KA_1^{-1}\tilde{C}^{-1}((\|x\|+\epsilon) + 2K(\|x\|+\epsilon) + 2\tilde{C}\epsilon)$$
$$\leq \ 2(1+\epsilon)K(2K+2)A_1^{-1}\tilde{C}^{-1}\|x\|.$$

So J_F is bounded.

Theorem V.C.1 considers operators from spaces with FDD. For operators from spaces without an FDD, we have the following corollaries.

Corollary V.C.4. *Let (u_i) be a normalized 1-subsymmetric basis for U. Let (v_i) be a normalized 1-subsymmetric basis for V and let (w_i) be a normalized subsymmetric basis for W. Suppose that (u_i) dominates (w_i) and (w_i) dominates (v_i). Let X be a separable reflexive Banach space which satisfies a lower-U-tree estimate and let Y be a separable reflexive Banach space which satisfies an upper-V-tree estimate. Then any bounded linear operator T from X into Y factors through a subspace of $Z_W(F)$, where $F = (F_n)$ is an FDD for some reflexive space Z.*

Proof. By Theorem 3.4 in [27], Y embeds into a reflexive space $\tilde{Y}$ with an FDD (G_n) which satisfies an upper-V-tree estimate. We use $\tilde{U}, \tilde{V}$ and $\tilde{W}$ to denote the closed linear spans of $(u_i^*), (v_i^*)$ and (w_i^*) respectively, where $(u_i^*), (v_i^*)$ and (w_i^*) are the biorthogonal functional of $(u_i), (v_i)$ and (w_i) respectively. Let T^* be the adjoint operator of T. Since the image of T^* is inside X^* which satisfies an upper-$\tilde{U}$-tree estimate (Corollary 3.3 in [27]), T^* satisfies an upper-$\tilde{U}$-tree estimate. $\tilde{Y}$ satisfies an upper-V-tree estimate, so $\tilde{Y}^*$ satisfies a lower-$\tilde{V}$-tree estimate. By Theorem V.C.1, T^* factors through $\tilde{Z}_{\tilde{W}}(\tilde{F}_n)$. By considering T^{**}, which is T, we conclude that T factors through a subspace of $Z_W(F)$.

Corollary V.C.5. *Let $1 < q \leq r \leq p < \infty$ and let X be a separable reflexive Banach space which satisfies a lower-ℓ_q-tree estimate. Let T be a bounded linear operator from X into Y which satisfies an upper-ℓ_p-tree estimate. Then T factors through a subspace of $(\sum F_n)_{l_r}$, where (F_n) is a sequence of finite dimensional spaces.*

Proof. By Theorem 2.1 in [26], X is a quotient of a reflexive space Z with an FDD which satisfies an lower-q-tree estimate. Let Q be a quotient map from Z onto X. Then it is easy to see that $T \circ Q$ satisfies an upper-p-tree estimate. By Theorem V.C.1, $\tilde{T} = T \circ Q$ factors through $(E_n)_{l_r}$, where (E_n) is a sequence of finite dimensional spaces. Let $\tilde{T}^*$ be the adjoint operator of $\tilde{T}$. Then $\tilde{T}^* = J \circ T^*$ factors through $(E_n^*)_{l_{r'}}$, where J is an embedding of X^* into Z^* and $\dfrac{1}{r'} + \dfrac{1}{r} = 1$. This implies that T^* factors through a subspace H of $(E_n^*)_{l_{r'}}$. By [18], H is also a quotient of some $(F_n)_{l_{r'}}$. By considering T^{**}, which is T, we deduce that T factors through a subspace of $(F_n^*)_{l_r}$.

Definition V.C.6. Let μ be a positive measure and let f be a scalar-valued μ-measurable function which is finite almost everywhere. The distribution function $m(\theta, f)$ is defined by

$$m(\theta, f) = \mu(x : |f(x)| > \theta).$$

Definition V.C.7. If f is a μ-measurable function, we denote by f^* its decreasing rearrangement, i.e.

$$f^*(t) = \inf\{\theta : m(\theta, f) \leq t\}.$$

The Lorentz space L_{pr} is defined as follows. We have $f \in L_{pr}, 1 \leq p \leq \infty$, if and only if

$$\|f\|_{L_{pr}} = (\int_0^\infty t^{\frac{1}{p}-1} f^*(t)^r dt)^{1/r} < \infty, 1 \leq r < \infty,$$

$$\|f\|_{L_{pr}} = \sup_t t^{\frac{1}{p}} f^*(t) < \infty, r = \infty.$$

Remark V.C.8. Let $1 \leq p \leq \infty$ and let $a_1 \geq a_2 \geq ... \geq a_n \geq ... \geq 0$. If we consider (a_i) as an element in l_{p1}, then the norm of (a_i) is computed in the following way:

$$\|(a_i)\|_{l_{p1}} = \sum i^{\frac{1}{p}-1} a_i.$$

Remark V.C.9. Let $1 \leq r < p \leq \infty$. Then the canonical basis of l_r dominates the canonical basis of l_{p1}.

Definition V.C.10. Let V be a Banach space with 1-unconditional normalized basis (v_i). For $0 < \gamma < 1$, we introduce the Tsirelson space $T(V, \gamma)$ associated to V and γ as follows. It is the completion of c_{00} under the norm $\|\cdot\|_{T(V,\gamma)}$, where

$$\|x\|_{T(V,\gamma)} = \max_{l \in \mathbf{N_0}} \{\|x\|_{l,T(V,\gamma)}\}, \forall x \in c_{00},$$

and the norms $\|\cdot\|_{l,T(V,\gamma)}$ on c_{00} are defined recursively. We put

$$\|x\|_{0,T(V,\gamma)} = \|x\|_\infty = \max_{i \in \mathbf{N}} \{|x_i|\},$$

and, assuming $\|\cdot\|_{l,T(V,\gamma)}$ has been defined, we put

$$\|x\|_{l+1,T(V,\gamma)} = \|x\|_{l,T(V,\gamma)} \bigvee \max_{n \in \mathbf{N}, n \leq A_1 < A_2 < ... < A_n} \{\gamma \| \sum_{i=1}^n \|P_{A_i}(x)\|_{l,T(V,\gamma)} v_i \|_V\},$$

where for $A, B \subset \mathbf{N}$ and $n \in \mathbf{N}$, $n \leq A$ means that $n \leq a$ for all $a \in A$, and $A < B$ means that $a < b$ for all $a \in A$ and $b \in B$. P_A, for $A \subset \mathbf{N}$, denotes the projection $\sum a_i e_i \mapsto \sum_{i \in A} a_i e_i$.

Theorem V.C.11. *Let $1 < q < p < \infty$ and let X be a separable reflexive Banach space which satisfies an asymptotic lower-ℓ_q-tree estimate. Let T be a bounded* *linear operator from X which satisfies an asymptotic upper-ℓ_p-tree estimate and let $q < r < p$. Then T factors through a subspace of $(F_n)_{l_r}$, where (F_n) is a sequence of finite dimensional spaces.*

To prove the theorem, we need the following lemmas. The first one is proved by E. Odell, Th. Schlumprecht and A. Zsak.

Lemma V.C.12. (Proposition 4.5 in [27]) *Let $1 < p < \infty$. For a separable reflexive Banach space X, the following are equivalent:*

a) X satisfies an asymptotic lower-ℓ_p-tree estimate.

b) There exists $\gamma \in (0, 1)$ such that X satisfies a lower-$T(\ell_p, \gamma)$-tree estimate.

Lemma V.C.13. *Let $1 < p < \infty$. Let (u_i) be the canonical basis for l_{p1} and let (v_i) be a normalized asymptotic ℓ_p basic sequence. Then (u_i) dominates (v_i).*

Proof. By definition, we need to prove that there is a $C > 0$ so that for any $(a_i) \subset \mathbf{R}$,

$$\| \sum a_i v_i \| \leq C \| \sum a_i u_i \|.$$

By scaling and a small perturbation, without loss of generality, we assume $0 < |a_i| \leq 1, \forall i \in \mathbf{N}$. Let (A_k) be the partition of $\mathbf{N}$ defined as

$$A_k = \{ i \in \mathbf{N} : \frac{1}{2^k} < |a_i| \leq \frac{1}{2^{k-1}} \}.$$

Let $I = \{ k \in \mathbf{N} : |A_k| > \frac{1}{2} \sum_{j<k} |A_j| \}$. There are two cases for I, finite or infinite. Since the proof when I is finite is essentially the same as when I is infinite, here we just give the proof for the case when I is infinite. Let $I = \{m_1, m_2, ...\}$, where $m_1 < m_2 < ....$ Denote $m_0 = 0$ and let $B_n = \cup_{k=m_n+1}^{m_{n+1}} A_k$. Let C be the asymptotic

ℓ_p constant for (v_i). Suppose $2^{t_n} \leq |A_{m_n}| < 2^{t_n+1}$. Then we have

$$\begin{aligned}
\|\sum_{i\in A_{m_n}} a_i v_i\| &\leq \frac{C}{2^{m_n-1}}(2^{\frac{t_n}{p}} + 2^{\frac{t_n-1}{p}} + ... + 1) \\
&\leq \frac{C}{2^{m_n-1}} \frac{2^{\frac{t_n}{p}}}{1-2^{-\frac{1}{p}}} \\
&= \frac{C}{1-2^{-\frac{1}{p}}} \frac{1}{2^{m_n}} |A_{m_n}|^{\frac{1}{p}}.
\end{aligned}$$

Noticing that for all $m_n < k < m_{n+1}$,

$$|A_k| \leq \frac{1}{2}(\frac{3}{2})^{k-m_n}|A_{m_n}|,$$

we get

$$\begin{aligned}
\sum_{k=m_n+1}^{m_{n+1}-1} \frac{C}{2^{k-1}}|A_k|^{\frac{1}{p}} &\leq \sum_{k=m_n+1}^{m_{n+1}-1} \frac{C}{2^{k-1}} \frac{1}{2^{\frac{1}{p}}} (\frac{3}{2})^{\frac{k-m_n}{p}} |A_{m_n}|^{\frac{1}{p}} \\
&= \sum_{k=m_n+1}^{m_{n+1}-1} \frac{1}{2^{m_n-1}} \frac{C}{2^{\frac{1}{p}}} \frac{(\frac{3}{2})^{\frac{k-m_n}{p}}}{2^{k-m_n}} |A_{m_n}|^{\frac{1}{p}}.
\end{aligned}$$

Now let

$$\tilde{C}_p = \frac{C}{1-2^{-\frac{1}{p}}} + \sum_{k=m_n+1}^{\infty} \frac{C}{2^{\frac{1}{p}}} \frac{(\frac{3}{2})^{\frac{k-m_n}{p}}}{2^{k-m_n}}.$$

We have

$$\begin{aligned}
\|\sum_{i\in B_n} a_i v_i\| &\leq \|\sum_{i\in A_{m_n}} a_i v_i\| + \sum_{k=m_n+1}^{m_{n+1}-1} \frac{C}{2^{k-1}}|A_k|^{\frac{1}{p}} \\
&\leq \frac{1}{2^{m_n}} \tilde{C}_p |A_{m_n}|^{\frac{1}{p}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\sum a_i u_i\| &\geq \sum_n \frac{1}{2^{m_n}}((\sum_{j=1}^{m_n}|A_j|)^{\frac{1}{p}} - (\sum_{j=1}^{m_n-1}|A_j|)^{\frac{1}{p}}) \\
&\geq \sum_n \frac{1}{2^{m_n}}(3^{\frac{1}{p}} - 2^{\frac{1}{p}})|A_{m_n}|^{\frac{1}{p}}
\end{aligned}$$

$$\geq \frac{3^{\frac{1}{p}} - 2^{\frac{1}{p}}}{\tilde{C}_p} \sum_n \| \sum_{i \in B_n} a_i v_i \|$$
$$\geq \frac{3^{\frac{1}{p}} - 2^{\frac{1}{p}}}{\tilde{C}_p} \| \sum a_i v_i \|.$$

This finishes the proof.

Lemma V.C.14. *Let $1 < p < \infty$ and let X be a separable reflexive Banach space. Let T be a bounded linear operator from X into a Banach space Y. For the following conditions,*

a) T satisfies an asymptotic upper-ℓ_p-tree estimate;

b) T satisfies an upper-l_{p1}-tree estimate;

we have a) implies b).

Proof. Since X is a separable reflexive Banach space, by Zippin's theorem [34], we can assume that X is a subspace of a reflexive space with an FDD (E_n). Let $C > 0$ be the constant associated with the asymptotic upper-ℓ_p-tree estimate. For $k \in \mathbf{N}$, let

$$\mathcal{A}^k = \{(x_i) \in S_X^\omega : \|T(\sum_{i=1}^k a_i x_i)\| \leq C(\sum_{i=1}^k |a_i|^p)^{1/p}, \forall (a_i)_{i=1}^k \subset \mathbf{R}\}.$$

Let $\epsilon > 0$ be so small that for all $k \in \mathbf{N}$,

$$\overline{\mathcal{A}_\epsilon}^k \subset \{(x_i) \in S_X^\omega : \|T(\sum_{i=1}^k a_i x_i)\| \leq 2C(\sum_{i=1}^k |a_i|^p)^{1/p}, \forall (a_i)_{i=1}^k \subset \mathbf{R}\}.$$

Let $(E_i^0) = (E_i)$. Applying Lemma V.C.2, we get a decreasing null sequence $\delta^1 = (\delta_i^1)$ and a blocking (E_i^1) of (E_i^0) so that if $(x_i) \subset S_X$ is a δ^1-skipped block sequence of (E_i^1), then (x_i) lies in $\overline{\mathcal{A}_\epsilon^1}$. By doing this procedure repeatedly, we obtain decreasing null sequences $\delta^k = (\delta_i^k)$ and blockings (E_i^k) of (E_i^{k-1}) so that if $(x_i) \subset S_X$ is a (δ_i^k)-skipped block sequence of (E_i^k), then (x_i) lies in $\overline{\mathcal{A}_\epsilon^k}$. Let $E_k^k = \bigoplus_{i=n_k}^{m_k} E_i$. And let

(F_i) be a blocking of (E_i) so that $F_k = \bigoplus_{i=n_k}^{n_{k+1}-1} E_i$. We can then choose a decreasing null sequence $\delta = (\delta_i)$ so that if $(x_i) \subset S_X$ is a δ-skipped block sequence of (F_i), then (x_i) is a basic sequence and any normalized block sequence $(z_i)_{i=1}^k$ of $(x_i)_{i=k}^\infty$ is a δ^k-skipped block sequence of (E_i^k). Hence, by Lemma V.C.13, (Tx_i) is dominated by the canonical basis of l_{p1}. This shows T satisfies an upper-l_{p1}-tree estimate.

Proof of Theorem V.C.11. By Lemma V.C.12, X satisfies an lower-$T(\ell_q, \gamma)$-tree estimate for some $0 < \gamma < 1$. By Lemma V.C.14, T satisfies an upper-l_{p1}-tree estimate. Since the canonical basis of $T(\ell_q, \gamma)$ dominates the canonical basis of l_r and the canonical basis of l_r dominates the canonical basis of l_{p1} when $1 < q \leq r \leq p < \infty$, by Theorem V.C.1, we get that T factors through a subspace of $(F_n)_{l_r}$, where $F = (F_n)$ is a sequence of finite dimensional spaces.

By Theorem V.C.11, we have

Corollary V.C.15. *Let $1 < q < p < \infty$ and let X be a reflexive asymptotic ℓ_q space. Let T be a bounded linear operator from X which satisfies an asymptotic upper-ℓ_p-tree estimate. Then T factors through a subspace of a space with a (p, q)-FDD.*

Remark V.C.16. Theorem V.C.11 and Corollary V.C.16 start with asymptotic conditions while end up with factorizations through subspaces of spaces with properties much stronger than asymptotic properties. This gives us some information on the relations between asymptotic ℓ_p spaces and $(F_n)_{l_r}$ spaces. However they do not tell us what happens when $p = q$.

The following theorem provides a result for a special case when $p = q$.

Theorem V.C.17. *Let $2 < p < \infty$. Let X be a separable reflexive asymptotic ℓ_p space. Let T be a bounded linear operator from X into L_p which satisfies an asymptotic upper-ℓ_p-tree estimate. Then T factors through ℓ_p.*

Proof. W. B. Johnson proved in [11] that for $p > 2$, a bounded linear operator T into L_p factors through ℓ_p if and only if T is compact when considered as an operator into L_2. So it is enough to show that T is compact as an operator into L_2. By Corollary 4.8 in [27], X embeds into a reflexive Banach space with an asymptotic ℓ_p FDD (E_n). Let (h_n) be the canonical Haar basis of L_2. If T is not compact as an operator into L_2, then there are a $\delta > 0$ and a normalized block sequence (x_i) with respect to (E_n) so that $(i_{p,2} \circ Tx_i)$ is essentially a block sequence with respect to (h_n) and $\|i_{p,2} \circ Tx_i\| > \delta, \forall i \in \mathbf{N}$, where $i_{p,2}$ is the formal identity map from L_p into L_2. This gives a contradiction since on the normalized weakly null tree $(x_A)_{A \in [\mathbf{N}]^{<\omega}}, x_A = x_{\max\{A\}}$, T does not satisfy an asymptotic upper-ℓ_p-tree estimate.

Remark V.C.18. Theorem V.C.17 holds even if we only assume that for every normalized weakly null sequence in X there is a subsequence the image of which under T satisfies an asymptotic upper-l_r estimate for some $2 < r < \infty$.

CHAPTER VI

SUMMARY

In this chapter, we summarize results proved in the dissertation.

A. Characterizations of subspaces of a Banach space with an unconditional basis

A Banach space X is said to have the unconditional tree property if for every normalized weakly null tree in X, there is a branch which is unconditional.

Theorem. *Let X be a separable reflexive space. Then X embeds into a reflexive space with an unconditional basis if and only if X has the unconditional tree property.*

B. Necessary and sufficient conditions for operators from L_p $(2 < p < \infty)$ to factor through ℓ_p

Let T be a bounded linear operator from a Banach space X. We say that T satisfies an upper-(C,p)-tree estimate if for every normalized weakly null tree in X, there is a branch (x_i) such that for all $(a_i) \subset \mathbf{R}$,

$$\|T(\sum a_i x_i)\| \leq C(\sum |a_i|^p)^{1/p}.$$

Theorem. *Let $2 < p < \infty$ and let T be a bounded linear operator from L_p. Then T factors through ℓ_p if and only if T satisfies an upper-(C,p)-tree estimate.*

C. Results on operators from reflexive spaces with asymptotic structures

Let $0 \leq p \leq \infty$. A reflexive Banach space X satisfies an asymptotic lower-ℓ_p-tree estimate if there exists a $0 < C < \infty$ so that for every $k \in \mathbf{N}$, every normalized weakly null tree of length k in X has a branch $(x_i)_{i=1}^k$ such that for all $(a_i) \subset \mathbf{R}$,

$$\|\sum_{i=1}^k a_i x_i\| \geq C^{-1}(\sum_{i=1}^k \|a_i\|^p)^{1/p}.$$

Let $0 \leq q \leq \infty$ and let T be a bounded linear operator from a Banach space X. T satisfies an asymptotic upper-ℓ_q-tree estimate if there exists a $0 < C < \infty$ so that for every $k \in \mathbf{N}$, every normalized weakly null tree of length k in X has a branch $(x_i)_{i=1}^k$ such that

$$\|\sum_{i=1}^k x_i\| \leq C(\sum_{i=1}^k \|x_i\|^q)^{1/q}.$$

Theorem. *Let $1 < q < p < \infty$ and let X be a separable reflexive Banach space which satisfies an asymptotic lower-ℓ_q-tree estimate. Let T be a bounded linear operator from X which satisfies an asymptotic upper-ℓ_p-tree estimate and let $q < r < p$. Then T factors through a subspace of $(F_n)_{l_r}$, where (F_n) is a sequence of finite dimensional spaces.*

At the end of the dissertation, we provide a serial of open problems in the structure theory of Banach spaces.

D. Problem on embedding theory

A Banach space X is uniformly convex if for every $\epsilon > 0$, there is a $\delta > 0$ so that for every $x, y \in S_X$ with $\|x - y\| > \epsilon$,

$$|||\frac{x+y}{2} - 1||| > \delta.$$

Let (E_i) be an FDD for X and let $1 \leq q \leq p \leq \infty$. We say that (E_i) is a (p, q) FDD if there is a $C > 0$ so that for every block sequence (x_n) of (E_n),

$$C^{-1}(\sum \|x_n\|^p)^{1/p} \leq \|\sum x_n\| \leq C(\sum \|x_n\|^q)^{1/q}.$$

In [26], E. Odell and Th. Schlumprecht proved that every uniformly convex Banach space embeds into a reflexive Banach space with an (p, q)-FDD and asked the following question.

Question VI.D.1. *Does every uniformly convex Banach space embed into an uniformly convex Banach space with an FDD?*

E. Problem on finite dimensional spaces

In the late 80's, S. J. Szarek and M. Talagrand showed that the Banach-Mazur distance between any n dimensional space and l_1^n or l_∞^n does not exceed $Cn^{7/8}$ where C is an absolute constant. In 1995, A. Giannopoulos [8] improved the upper bound to $Cn^{5/6}$. But these exponents 7/8 and 5/6 do not seem to be natural.

Question VI.E.1 *Find the essential upper bound for the Banach-Mazur distance from any n-dimensional Banach space to l_1 or l_∞.*

F. Problems in nonlinear functional analysis

Many natural operators between Banach spaces turn out to nonlinear. In general , it is much harder to deal with nonlinear operators than linear operators. So a very important question in nonlinear functional analysis is to study the conditions under which a certain class of nonlinear operators can be replaced by linear operators. The classes of nonlinear operators we are interested in are so called Lipschitz continuous and uniformly continuous operators.

An operator α from a Banach space X into a Banach space Y is Lipschitz if there is a $C > 0$ so that for any $x_1, x_2 \in X$,

$$\|\alpha(x_1) - \alpha(x_2)\| \leq C\|x_1 - x_2\|.$$

It is called uniformly continuous if for any $\epsilon > 0$, there is a $\delta > 0$ so that for any $x_1, x_2 \in X$ with $\|x_1 - x_2\| < \delta$,

$$\|\alpha(x_1) - \alpha(x - 2)\| < \epsilon.$$

Two Banach spaces X and Y are Lipschitz homeomorphic if there is a one-to-one Lipschitz operator α from X onto Y and α^{-1} is also Lipschitz continuous. In this case, we say that α is a Lipschitz homeomorphism. Similarly, X and Y are uniformly homeomorphic if there is a one-to-one uniformly continous operator α from X onto Y and α^{-1} is also uniformly continuous. And we call α a uniform homeomorphism. The following open problem is famous.

Question VI.F.1. *If X and Y are two separable Banach space which are Lipschitz homeomorphic, are they linearly isomorphic?*

In other words, one wants to know whether a Lipschitz homeomorphism between two separable Banach spaces can be replaced by a linear isomorphism.

A uniformly continuous operator α from X to Y is called a uniform quotient if it is onto and for each $\epsilon > 0$, there is a $\delta > 0$ so that for every $x \in X$, $B_\delta(\alpha(x)) \subset \alpha(B_\epsilon(x))$. If in addition α is Lipschitz continuous and δ can be chosen to be linearly dependent on ϵ, it is called a Lipschitz quotient.

In [3], S. Bates, W. B. Johnson, J. Lindenstrauss, D. Preiss and G. Schechtman proved that any uniform quotient of $L_p (1 < p < \infty)$ is isomorphic to a linear quotient of L_p. In [9], G. Godefroy, N. J. Kalton and G. Lancien proved that if X is Lipschitz homeomorphic to a linear quotient of $\ell_p(2 < p < \infty)$, then X is linearly isomorphic to a linear quotient of ℓ_p. In [31], L. Randrianarivony proved the same result for $1 < p < 2$.

Question VI.F.2. *Let X be a Lipschitz quotient of $\ell_p(1 < p < \infty)$. Is X isomorphic to a linear quotient of ℓ_p?*

REFERENCES

[1] D. Alspach, R. Judd, E. Odell, *The Szlenk index and local ℓ_1-indices*, Positivity 9 (2005), 1-44.

[2] S. Banach, *Theorie des operations lineaires*, Warszawa, 1932.

[3] S. Bates, W. B. Johnson, J. Lindenstrauss, D. Preiss and G. Schechtman, *Affine approximation of Lipschitz functions and nonlinear quotients*, Geom. Funct. Anal. 9 (1999), no. 6, 1092–1127.

[4] W. J. Davis, T. Figiel, W. B. Johnson and A. Pełczyński, *Factoring weakly compact operators*, Journal of Functional Analysis, 17 (1974), No. 3, 311-327.

[5] M. Feder, *On subspaces of spaces with an unconditional basis and spaces of operators*, Illinois Journal of Mathematics, 24 (1980), 196-205.

[6] M. Fabian, P. Habala, P. Hajek, V. M. Santalucia, J. Pelant and V. Zizler, *Functional Analysis and Infinite-Dimensional Geometry*, Springer-Verlag, New York Berlin Heidelberg, 2001.

[7] T. Figiel, W. B. Johnson and L. Tzafriri, *On Banach lattices and spaces having local unconditional structure, with applications to Lorentz function spaces*, Journal of Approximation Theory 13 (1975), 395-412.

[8] A. A. Giannopoulos, *A note on the Banach-Mazur distance to the cube*, Geometric aspects of functional analysis (Israel, 1992–1994), 67–73, Oper. Theory Adv. Appl., 77, Birkhauser, Basel, 1995.

[9] G. Godefroy, N. J. Kalton and G. Lancien, *Szlenk indices and uniform homeomorphisms*, Transactions of the American Mathematical Society, 353 (2001), 3895-3918.

[10] R. Haydon, E. Odell and Th. Schlumprecht, *Small subspaces of L_p*, preprint.

[11] W. B. Johnson, *Operators into L_p which factor through ℓ_p*, Journal of the London Mathematical Society (2), 14 (1976), 333-339.

[12] W. B. Johnson, *On quotients of L_p which are quotients of ℓ_p*, Compositio Mathematica, 34 (1977), Fasc. 1, 69-89.

[13] W. B. Johnson, *Seminaire D ' analyse Fonctionnelle, Operators into L_p which factor through ℓ_p*, Ecole Polytechnique, Centre De Mathematiques, No. XVII, (1980).

[14] W. B. Johnson, E.Odell, *Subspaces of L_p which embed into ℓ_p*, Compositio Mathematica, 28 (1974), Fasc. 1, 37-49.

[15] W. B. Johnson, H. P. Rosenthal, *On ω^*-basic sequences and their applications to the study of Banach spaces*, Studia Math., 43, (1972), 77-92.

[16] W. B. Johnson, B. Zheng, *A characterization of quotients and subspaces of reflexive spaces with unconditional bases*, To appear in Duke Mathematical Journal.

[17] W. B. Johnson, M. Zippin, *On subspaces of quotients of $(\sum G_n)_{\ell_p}$ and $(\sum G_n)_{c_0}$*, Israel Journal of Mathematics, 13 (1972), 311-316.

[18] W. B. Johnson, M. Zippin, *Subspaces and quotient spaces of $(\sum G_n)_{\ell_p}$ and $(\sum G_n)_{c_0}$*, Israel Journal of Mathematics, 17, (1974), 50-55.

[19] N. J. Kalton, *On subspaces of* c_0 *and extension of operators into* $C(K)$*-spaces*, Quarterly Journal of Mathematiics 52 (2001), no. 3, 313-328.

[20] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I, sequence spaces*, Springer-Verlag, New York, 1977.

[21] B. Maurey, V. D. Milman and N. Tomczak-Jaegermann, *Asymptotic infinite-dimensional theory of Banach spaces*, Oper. Theory: Adv. Appl. 77 (1994), 149175.

[22] V.D. Milman, N. Tomczak-Jaegermann, *Asymptotic* ℓ_p *spaces and bounded distortions*, eds. Bor-Luh Lin and W.B. Johnson, Contemp. Math. 144 (1993), 173195.

[23] E. Odell, *Applications of Ramsey theorems to Banach space theory*, in: Notes in Banach spaces, ed. H. E. Lacey, Univ. of Texas Press, Austin,TX (1980), 379-404.

[24] E. Odell, *On quotients of Banach spaces having shrinking unconditional bases*, Illinois Journal of Mathematics, 36 (1992), No. 4, 681-695.

[25] E. Odell, Th. Schlumprecht, *Trees and branches in Banach spaces*, Transactions of the American Mathematical Society 354 (2002), No. 10, 4085-4108.

[26] E. Odell, Th. Schlumprecht, *A universal reflexive space for the class of uniformly convex Banch spaces*, Math. Ann. 335 (2006), no. 4, 901-916.

[27] E. Odell, Th. Schlumprecht, A. Zsak, *On the structure of asymptotic* ℓ_p *spaces*, preprint.

[28] A. Pełczyński, *Projections in certain Banach spaces*, Studia Math. 19 (1960), 209-228.

[29] A. Pełczyński and P. Wojtaszczyk, *Banach spaces with finite dimensional expansions of identity and universal bases of finite dimensional subspaces*, Studia Math., 40 (1971), 91-108.

[30] H. P. Rosenthal, *A characterization of Banach spaces containing* ℓ_1, Proc. Natl. Acad. Sci. USA, 71 (1974), 2411-2413.

[31] L. Randrianarivony, *Nonlinear classification of Banach spaces*, Ph.D. dissertation, Texas A&M University, 2005.

[32] B. Zheng, *On operators which factor through* ℓ_p *or* c_0, Studia Math., 176 (2006), No. 2, 177-190.

[33] B. Zheng, *On operators from separable reflexive spaces with asymptotic structure*, preprint.

[34] M. Zippin, *Banach spaces with separable duals*, Transactions of the American Mathematical Society, 310 (Nov., 1988), No. 1 , 371-379.

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